ON RAMANUJAN BIGRAPHS AND GENERAL RAMANUJAN GRAPHS

SHAI EVRA, BROOKE FEIGON, KATHRIN MAURISCHAT, ORI PARZANCHEVSKI

Abstract. In this paper, we give an overview of new results that define and explicitly construct Ramanujan Cayley biregular bipartite graphs. We also study the extremal combinatorial properties of these graphs. This parallels the work of Lubotzky, Phillips and Sarnak on regular Ramanujan Cayley graphs, with several interesting differences. Furthermore, this work also proposes a stronger definition of Ramanujan graphs than has been used in the past, which opens the door to future studies.

Keywords: Ramanujan graphs, Cayley bigraphs, non-backtracking spectrum, pseudorandomness, simply-transitive lattices, Ramanujan conjecture, automorphic representations of $U(3)$

2020 Mathematics Subject Classification: Primary: 11F70 Number Theory (math.NT); Secondary: 05C48 Combinatorics (math.CO)

1. Introduction

Regular Ramanujan graphs were defined by Lubotzky, Phillips and Sarnak in their seminal paper [LPS88]. These graphs stand out as optimal *expanders* – sparse graphs with strong connectivity, which have numerous applications in mathematics, computer science, and other areas. The main result of [LPS88] is an explicit construction of infinite families of regular Ramanujan graphs, which are furthermore Cayley graphs, now called LPS graphs.

A natural question that arises is how to define and how to construct non-regular Ramanujan graphs. These questions were touched upon by many researchers, but remain unresolved for the most part. In this article, we give an overview of our work on Ramanujan bigraphs [EFMP23] that will appear elsewhere. This work begins a systematic study of the problem, building a complete theory for the case of biregular bipartite graphs (*bigraphs*, for short). We study several definitions of Ramanujan graphs and highlight the strongest one, which yields the strongest expansion, and for which there had not been previously an explicit construction. Our main goal is the construction of Ramanujan bigraphs that have an explicit Cayley-like description à la LPS. The LPS construction uses Deligne's proof of the Ramanujan-Petersson conjecture for $PGL_2/\mathbb{Q}$, and a major obstacle in generalizing their result to other settings is that the (naive) Ramanujan conjecture is false for other groups. The explicit construction of our Ramanujan bigraphs is achieved by proving two separate results in the theories of arithmetic groups and automorphic representations: a construction of p-arithmetic lattices in $PU_3(\mathbb{Q}_p)$ that act simply-transitively on the hyperspecial vertices of the Bruhat-Tits trees, and a study of the automorphic representations which violate the Naive Ramanujan Conjecture for the algebraic group $PU_3/\mathbb{Q}$. In addition, we study the combinatorial properties of Ramanujan bigraphs and compare and contrast them to the combinatorial properties of Ramanujan graphs. We also present additional applications of our work to quantum golden gates and other topics.

2. Ramanujan Cayley graphs and bigraphs

A connected k-regular graph X is a *Ramanujan graph* if every eigenvalue λ of the adjacency matrix of X satisfies either $|\lambda| = k$ (these are the "trivial eigenvalues") or $|\lambda| \leq 2\sqrt{k-1}$. In the work of LPS $k = p+1$, where p is a prime.

More generally, we call a graph X *weakly Ramanujan* if all non-trivial eigenvalues of its adjacency matrix are bounded by the spectral radius of its universal covering tree. We say X is *adj-Ramanujan* if every nontrivial eigenvalue of the adjacency matrix of X *belongs to* the adjacency spectrum of its universal covering tree. Finally, the "non-backtracking" (NB) operator $B = B_X$ of a graph X acts on functions on the directed edges in X by

$$(Bf)(v \to u) = \sum_{v \neq w \sim u} f(u \to w). \tag{2.1}$$

We say a graph is *NB-Ramanujan* if the nontrivial spectrum of B_X is contained in the spectrum of B acting on the universal covering tree of X. For bigraphs, but not in general, this is equivalent to the same property for *every* geometric operator on X, that is, operators that commute with the automorphisms of the universal cover, and also equivalent to the graph satisfying the Riemann hypothesis. This is why we call a bigraph that satisfies these equivalent definitions *Ramanujan*. More generally, we call a graph *Ramanujan* if it satisfies the strongest condition listed above, that is, the spectral condition for every geometric operator. We note that for the LPS case of regular graphs the notions of weakly Ramanujan, adj-Ramanujan and Ramanujan coincide, but they do not coincide in the bigraph or more general cases. In [MSS15], the authors prove the existence of infinite families of *weakly* Ramanujan bigraphs of any degrees.

More precisely, for a bigraph X with valency $(K+1, k+1)$, we distinguish the left from the right vertices by $K > k$. In a profound but essentially elementary study we show that for bigraphs X the spectrum of the adjacency operator A_X explicitly determines the large part of the spectrum of the non-backtracking operator B_X, the remaining part being determined by cycles in X. The concise knowledge of the spectra for bigraphs reveals that the Riemann hypothesis on the Ihara zeta function of X is equivalent to the NB-Ramanujan property. And, in turn, the latter is equivalent to the adj-Ramanujan property together with the vanishing of the kernel of A_X restricted to the left vertices. In contrast, the equivalence to the full Ramanujan property is a consequence of the considerations of the Iwahori Hecke algebra.

2.1. Cayley bigraphs.

In the work of [LPS88] they made their constructions explicit by showing that their graphs are Cayley graphs. However, Cayley graphs are always regular graphs, so we need a new notion that allows for an explicit description of bigraphs. We keep a group structure on the left vertices and define the right vertices by a certain equivalence relation that comes from a partition of a set of group generators which must satisfy certain properties to define a bigraph.

Definition. Let G be a group, $S^1, \dots, S^{K+1} \subseteq G$ and $S = \bigsqcup S^i$ such that $|S^i| = k, 1 \leq i \leq K+1$. Assume $S = S^{-1}$, $1 \notin S$ and if $s, t \in S^i$ then either $s = t$ or $s^{-1}t$ and $s^{-1} \in S^j$ for some j. The *Cayley bigraph* $CayB\left(G, \left\{S^i\right\}\right)$ is a $(K+1, k+1)$-bigraph $(L \sqcup R, E)$ defined by $L = G$, $R = {}^{G \times \{1,\dots,K+1\}}/_{\sim}$, where

$$(g,i) \sim (h,j) \quad \Longleftrightarrow \quad \begin{array}{ll} \text{either} & (g,i) = (h,j), \\ \text{or} & g^{-1}h \in S^i \text{ and } h^{-1}g \in S^j, \end{array}$$

and $E = \{\{g, [g,i]\} \mid g \in G, i \in \{1, \dots, K+1\}\}$, where $[g,i]$ is the equivalence class of (g,i).

By construction, G acts geometrically on the Cayley bigraph, the action on the left vertices being simply transitive. We also use the notion of a *Schreier bigraph*, in which the left vertices are given by a G-set instead of G itself, and for which similar axioms hold.

3. Main results

Theorem. *Let p and q be primes with $p \equiv 2 \pmod 3$, $q \notin \{3, p\}$, and $\omega = \frac{-1+\sqrt{-3}}{2}$. Let*

$$S_p := \left\{ g \in M_3\left(\mathbb{Z}\left[\omega\right]\right) \;\middle|\; \begin{matrix} g^* g = p^2 I, \ g \text{ is not scalar}, \\ g \equiv \left(\begin{smallmatrix} 1 & * & * \\ * & 1 & * \\ * & * & 1 \end{smallmatrix}\right) \pmod 3 \end{matrix} \right\}, \tag{3.1}$$

and let $S_p = \bigsqcup_i S_p^i$ be the partition induced by the equivalence relation

$$g \sim h \text{ if and only if } g^* h \in p M_3(\mathbb{Z}[\omega]).$$

Denote

$$\mathbf{G}_q := \begin{cases} PSL_3\left(\mathbb{F}_q\right) & q \equiv 1 \pmod 3 \\ PSU_3\left(\mathbb{F}_q\right) & q \equiv 2 \pmod 3, \end{cases}$$

and

$$S_{p,q}^i := S_p^i \pmod q \overset{(\star)}{\subseteq} \mathbf{G}_q$$

where $(\star)$ implies mapping ω to a root of $x^2 + x + 1$ in $\mathbb{F}_q$ or in $\mathbb{F}_{q^2}$ according to $q \pmod 3$. The Cayley bigraphs

$$X_{\mathscr{E}}^{p,q} = CayB\left(\mathbf{G}_q, \left\{S_{p,q}^i\right\}_i\right)$$

satisfy:

(1) $X_{\mathscr{E}}^{p,q}$ is an adj-Ramanujan $(p^3+1, p+1)$-regular bigraph, with left side of size $|\mathbf{G}_q| \approx q^8/3$.
(2) $X_{\mathscr{E}}^{p,q}$ is not fully Ramanujan. In particular, $\pm i p^{3/2}$ are non-trivial eigenvalues of $B_{X_{\mathscr{E}}^{p,q}}$, but are not in the spectrum of B acting on the covering tree of $X_{\mathscr{E}}^{p,q}$.
(3) The family $\left\{X_{\mathscr{E}}^{p,q}\right\}_q$ satisfies the Sarnak-Xue density hypothesis. In fact, $\pm i p^{3/2}$ are the unique non-trivial eigenvalues of $B_{X_{\mathscr{E}}^{p,q}}$ that are not contained in the spectrum of B acting on the covering tree of $X_{\mathscr{E}}^{p,q}$, and for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ (not depending on p and q) such that the multiplicities of $\pm i p^{3/2}$ in $\operatorname{Spec} B_{X_{\mathscr{E}}^{p,q}}$ are bounded by $C_\varepsilon \left|X_{\mathscr{E}}^{p,q}\right|^{3/8+\varepsilon}$.
(4) The group $\mathbf{G}_q$ acts on the set

$$Y_q := \begin{cases} \mathbb{P}^2(\mathbb{F}_q) & q \equiv 1 \pmod 3 \\ \left\{ v \in \mathbb{P}^2(\mathbb{F}_q[\omega]) \,\middle|\, v^* \cdot v = 0 \right\} & q \equiv 2 \pmod 3 \end{cases},$$

and the Schreier bigraphs $Y_{\mathscr{E}}^{p,q} = SchB(Y^q, \left\{S_{p,q}^i\right\}_i)$ are (fully) Ramanujan.
(5) The girth of $X_{\mathscr{E}}^{p,q}$ is larger than $2\log_p q$.
(6) The family $\left\{X_{\mathscr{E}}^{p,q}\right\}_q$ exhibits bounded cutoff: the non-backtracking random walk on $X_{\mathscr{E}}^{p,q}$ goes from $(1-\varepsilon)$-mixing to ε-mixing (in total-variation distance) in a number of steps which do not depend on q.
(7) (Diameter) For small enough $\varepsilon > 0$, and $\ell \geq \frac{1}{2}\log_p |E_{X_{\mathscr{E}}^{p,q}}| + 2\log_p\left(\frac{1}{\varepsilon}\right) + 3$, for any $e \in E_{X_{\mathscr{E}}^{p,q}}$ we have

$$\left| \left\{ e' \in E_{X_{\mathscr{E}}^{p,q}} \;\middle|\; \begin{matrix} \text{there is a non-backtracking path} \\ \text{of length } \ell \text{ from } e \text{ to } e' \end{matrix} \right\} \right| \geq (1-\varepsilon)\,|E_{X_{\mathscr{E}}^{p,q}}|.$$

Furthermore, for any two directed edges e_1, e_2 *in* $X_{\mathscr{E}}^{p,q}$ *there is a non-backtracking path from* e_1 *to* e_2 *of length at most* $\log_p |E_{X_{\mathscr{E}}^{p,q}}| + 10$.

We should stress that showing that the graphs $X_{\mathscr{E}}^{p,q}$ are *not* Ramanujan is highly nontrivial, and is in fact the first explicit example of combinatorial non-Ramanujan objects arising as congruence quotients of Bruhat-Tits buildings. This demonstrates that unlike the common expectation for the Riemann hypothesis in number theory, its p-adic (or graph-theoretic) analogue can indeed fail. Items (3), (6) and (7) in our theorem carry out Sarnak's "density philosophy", which conjectures that when the Riemann hypothesis fails, it can be replaced by a weaker density hypothesis that yields similar results.

Our results are deduced from a construction in the projective unitary group $PU_3\left(\mathbb{Q}(\omega), \left(\begin{smallmatrix} & & 1 \\ & 1 & \\ 1 & & \end{smallmatrix}\right)\right)$. We also prove an analogue of the theorem above for constructions in the following groups:

- $PU_3\left(\mathbb{Q}(\lambda), \left(\begin{smallmatrix} 3 & \overline{\lambda} & \overline{\lambda} \\ \lambda & 3 & \overline{\lambda} \\ \lambda & \lambda & 3 \end{smallmatrix}\right)\right)$, where $\lambda = \frac{-1+\sqrt{-7}}{2}$, inspired by [Mum79]),
- $PU_3\left(\mathbb{Q}(\eta), \left(\begin{smallmatrix} 10 & -2(\eta+2) & \eta+2 \\ -2(\bar{\eta}+2) & 10 & -2(\eta+2) \\ \bar{\eta}+2 & -2(\bar{\eta}+2) & 10 \end{smallmatrix}\right)\right)$, where $\eta = \frac{1-\sqrt{-15}}{2}$, inspired by [CMSZ93a] and [CMSZ93b].

In these two cases, we furthermore prove that the infinite families of Cayley bigraphs obtained are fully Ramanujan and not only adj-Ramanujan.

3.1. **Combinatorial results.** Regular Ramanujan graphs have a number of desirable properties. We answer naturally arising questions about generalizations for bigraphs by establishing and analyzing similar properties. These include the following:

- We present a pseudorandomness characterization of Ramanujan bigraphs and develop a natural and more general notion of biexpanders for which Ramanujan bigraphs are indeed optimal.
- Concerning sparsification, we observe that this notion of biexpanders does not imply that biexpanders are good sparsifiers for the complete bipartite graphs anymore, but that there is a possible interpretation of them sparsifying geometric line-plane graphs.
- We show that random walks on our bigraphs exhibit Diaconis' cutoff phenomenon (see part (6) of the above theorem) at the optimal time $\frac{1}{2}\log_p|E_X|$ with a logarithmic window size $\frac{3}{2}\log_p \log|E_X|$. This is a consequence of the Sarnak-Xue density hypothesis.
- LPS Ramanujan graphs are known to have large girth, and part (5) of the Theorem gives a logarithmic lower bound for the girth of the studied bigraphs, which plays a part in the proof of bounded cutoff for them.

4. Lattices and automorphic representations of unitary groups in three variables

4.1. **Simply transitive lattices.** From an arithmetic point of view, there are two kinds of projective unitary groups PU_3. First, those arising from an involution of the second kind on a division algebra and second, the ones given by such an involution on a matrix group. For both kinds, one considers the local Bruhat-Tits buildings at an inert prime p, which are

$(p^3+1, p+1)$-regular trees. For the first kind of group, if they are compact at infinity, the quotients of these trees by congruence subgroups give Ramanujan bigraphs due to the temperedness of their automorphic spectrum (see [BC11, HT01]). But due to their origin, these graphs are very difficult to realize explicitly (see [BFG+15]).

In contrast, the quotient graphs of Bruhat-Tits trees coming from a matrix group PU_3 can have non-tempered automorphic spectra. As we use this approach, a concise study of the spectrum is necessary to establish the Ramanujan property, as outlined in Section 4.2. Apart from that, the main idea then is to give lattices in $PU_3(\mathbb{Q}_p)$ which act *simply transitively* on the hyperspecial (i.e. left) vertices of the Bruhat-Tits tree in order to gain a construction by Cayley bigraphs as above. However, such lattices are rare and we altogether give four families: three arising from the groups mentioned in Section 3, and one given by [EP22]. These groups are all of class number one, and we use a Mass formula to show that certain sublattices of the $\mathbb{Z}\left[\frac{1}{p}\right]$-valued points give rise to simply transitive lattices.

4.2. **Representation theory and the Ramanujan conjecture.** In order to prove that our Cayley bigraphs are Ramanujan, we first explore how to translate the relevant conditions from the spectral theory of a quotient of the Bruhat-Tits tree to the representation theory of $PU_3(\mathbb{Q}_p)$. For congruence arithmetic lattices, we then translate this local condition to automorphic representations. Let the graph $X = \Lambda\backslash\mathcal{B}$, where $\mathcal{B}$ is the Bruhat-Tits tree of $PU_3(\mathbb{Q}_p)$ and Λ is a congruence arithmetic lattice. Let K' be a compact subgroup of $PU_3(\mathbb{A})$ corresponding to Λ. We show that X is Ramanujan if for every K'-spherical automorphic representation π of PU_3, π_p is either one-dimensional or tempered, and establish a parallel criterion for being adj-Ramanujan.

Next we study the automorphic representations of PU_3 under certain compact subgroups to get results about the existence of both Ramanujan and non-Ramanujan bigraphs. As mentioned in the introduction, to prove that the LPS graphs are Ramanujan, the Ramanujan-Petersson Conjecture for $PGL_2/\mathbb{Q}$ is used, that is, that for cuspidal automorphic irreducible unitary representations $\pi = \otimes_v \pi_v$ of $PGL_2(\mathbb{A})$, π_v is tempered for each v. However, the Naive Ramanujan Conjecture does *not* hold for $PU_3/\mathbb{Q}$. Indeed, representations that are locally non-tempered do occur. We address this obstacle in two steps. First we use the work of [Rog90], [Shi11] and others to deduce the Generalized Ramanujan Conjecture for the definite form of PU_3, that is, that *generic* cuspidal automorphic irreducible unitary representations are locally tempered. This is enough to conclude that all of our graphs are adj-Ramanujan.

Next, we use Rogawski's classification of the automorphic spectrum of U_3 [Rog90, Rog92] to study the levels of the non-tempered representations via their multiplicities and epsilon factors, the non-vanishing of p-adic integrals, conductors of Hecke characters, and depth preservation of the theta correspondence. Through this work, we are able to find specific arithmetic congruence lattices in $PU_3(\mathbb{Q}_p)$ where only generic representations occur in their quotients, and we use these lattices to construct infinite families of Ramanujan bigraphs. We also use the failure of the Naive Ramanujan Conjecture along with our work on the levels of non-tempered representations to find arithmetic congruence lattices where infinite-dimensional non-generic representations do occur in the quotient, and use these lattices to construct infinite families of non-Ramanujan graphs, which are still weakly and adj-Ramanujan. In addition, we prove the Sarnak-Xue Density Hypothesis [SX91] for 3×3 unitary groups associated to a definite form, giving an upper bound on the multiplicity of non-tempered representations of a given level.

5. Further applications and future directions

Finally, we present additional applications of our work: golden and super golden gates for PU_3, Ramanujan and non-Ramanujan Cayley complexes of type $\widetilde{A}_2$, optimal strong approximation for p-arithmetic subgroups of PSU_3, and vanishing of the first Betti numbers of Picard modular surfaces.

Our study naturally suggests further research, for example, what are Ramanujan graphs and what are their combinatorial properties beyond the regular and biregular cases? Do they even exist? Going beyond PGL_2 and PU_3, what implications for Ramanujan complexes does the failure of the Naive Ramanujan Conjecture for other groups have?

Acknowledgement. This research was supported by the IAS Summer Collaborators Program, and the authors are grateful for the hospitality of the Institute for Advanced Study. They are also thankful to the American Institute of Mathematics, where this project was promoted during a workshop on Arithmetic Golden Gates. The authors were supported by NSF grant DMS-1638352 and ISF grant 1577/23 (S.E.), Simons Foundation Collaboration Grant 635835 (B.F.), and ISF grant 2990/21 (O.P.).

References

[BC11] C. Ballantine and D. Ciubotaru, *Ramanujan bigraphs associated with* SU(3) *over a p-adic field*, Proc. Amer. Math. Soc. **139** (2011), no. 6, 1939–1953.

[BFG+15] C. Ballantine, B. Feigon, R. Ganapathy, J. Kool, K. Maurischat, and A. Wooding, *Explicit construction of Ramanujan bigraphs*, Women in numbers europe, 2015, pp. 1–16.

[CMSZ93a] D. I Cartwright, A. M. Mantero, T. Steger, and A. Zappa, *Groups acting simply transitively on the vertices of a building of type $\tilde{A}_2$, I*, Geometriae Dedicata **47** (1993), no. 2, 143–166.

[CMSZ93b] ———, *Groups acting simply transitively on the vertices of a building of type $\tilde{A}_2$, II: The cases $q=2$ and $q=3$*, Geometriae Dedicata **47** (1993), no. 2, 167–223.

[EFMP23] S. Evra, B. Feigon, K. Maurischat, and O. Parzanchevski, *Ramanujan bigraphs*, arXiv preprint arXiv:2312.06507 (2023).

[EP22] S. Evra and O. Parzanchevski, *Ramanujan complexes and Golden Gates in PU(3)*, Geometric and Functional Analysis **32** (2022), 193–235.

[HT01] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, 2001.

[LPS88] A. Lubotzky, R. Phillips, and P. Sarnak, *Ramanujan graphs*, Combinatorica **8** (1988), no. 3, 261–277.

[MSS15] A. Marcus, D. A. Spielman, and N. Srivastava, *Interlacing families I: Bipartite Ramanujan graphs of all degrees*, Annals of Mathematics **182** (2015), 307–325.

[Mum79] D. Mumford, *An algebraic surface with K ample,* $(K^2) = 9$, $p_g = q = 0$, American Journal of Mathematics **101** (1979), no. 1, 233–244.

[Rog90] J. D. Rogawski, *Automorphic representations of unitary groups in three variables*, Annals of Mathematics Studies, vol. 123, Princeton University Press, Princeton, NJ, 1990.

[Rog92] ———, *Analytic expression for the number of points mod p*, The zeta functions of Picard modular surfaces, 1992, pp. 65–109.

[Shi11] S. W. Shin, *Galois representations arising from some compact Shimura varieties*, Annals of Mathematics **173** (2011), no. 3, 1645–1741. MR2800722

[SX91] P. Sarnak and X. Xue, *Bounds for multiplicities of automorphic representations*, Duke Mathematical Journal **64** (1991), no. 1, 207–227.

Shai Evra: Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Israel

Email address: shai.evra@mail.huji.ac.il

Brooke Feigon: Department of Mathematics, The City College of New York, CUNY, New York, NY 10031

Email address: bfeigon@ccny.cuny.edu

Kathrin Maurischat: Lehrstuhl für Algebra und Darstellungstheorie, RWTH Aachen University, D-52062 Aachen

Email address: kathrin.maurischat@art.rwth-aachen.de

Ori Parzanchevski: Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Israel

Email address: ori.parzan@mail.huji.ac.il