A RELATIVE TRACE FORMULA FOR PGL(2) IN THE LOCAL SETTING

BROOKE FEIGON

In memory of Jonathan Rogawski

Abstract. We develop the local Kuznetsov trace formula on a unitary group in two variables for an unramified quadratic extension of local, non-Archimedean fields E/F and compare it to a local relative trace formula on $\mathrm{PGL}(2,E)$. To define the local distributions for the relative trace formula, we define a regularized local period integral and prove that it is a $\mathrm{PGL}(2,F)$-invariant linear functional. By comparison of the two local trace formulas, we get an equality between a local $\mathrm{PGL}(2,F)$-period and local Whittaker functionals.

Contents

1. Introduction

Base change is an important type of functoriality which is useful in the study of automorphic forms by relating automorphic representations on different groups. Hervé Jacquet shed light on a new technique for attacking certain cases of Robert Langlands' important functoriality conjectures by comparing the relative and Kuznetsov trace formulas in the global setting. Jacquet's comparison of trace formulas leads to global identities that characterize the image of the base change map associating automorphic representations of a unitary group for a quadratic extension of number fields E/F to automorphic representations of $\mathrm{GL}(2,\mathbb{A}_E)$ in terms of distinguished representations. While Jacquet's global identities factor, they do not give unique local identities.

This paper uses techniques of James Arthur to define and develop a local Kuznetsov trace formula on U(2) and a local relative trace formula on GL(2). Both local trace formulas are expanded geometrically in terms of orbital integrals and spectrally in terms of local Bessel distributions and local relative Bessel distributions. The latter involve regularized local period integrals. We then carry out Jacquet's comparison in the local setting by relating these two local trace formulas for matching functions. This comparison yields identities between local Bessel distributions for automorphic representations on U(2) and local relative Bessel distributions for automorphic representations on GL(2).

Before we describe more precisely the local relative trace formula developed in this paper, let us recall the relative trace formula for GL(2). Take E/F to be a quadratic extension of number fields and $\mathbb{A}_F$ to be

Date: October 10, 2012.
2010 *Mathematics Subject Classification.* 11F70, 22E50.
The author was supported by NSF grant DMS-1201446.

the adeles of F. Let ψ' be a character on $F\backslash\mathbb{A}_F \cong N(F)\backslash N(\mathbb{A}_F)$ where N is the upper triangular unipotent matrices of $\mathrm{GL}(2)$. Let $\psi = \psi' \circ \mathrm{tr}_{E/F}$.

A cuspidal automorphic representation π of $\mathrm{GL}(2,\mathbb{A}_E)$ is *distinguished* by $\mathrm{GL}(2,\mathbb{A}_F)$ if there exists a $\phi \in V_\pi$, the vector space associated to π, such that the *period integral*, $P(\phi)$, is nonzero:

$$P(\phi) := \int_{\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A}_F)} \phi(h)dh \neq 0.$$

For π' a cuspidal automorphic representation of the quasi-split unitary group $U(2,\mathbb{A}_F)$ and $\phi' \in V_{\pi'}$, let

$$W(\phi') = \int_{N(F)\backslash N(\mathbb{A}_F)} \phi'(n)\overline{\psi'(n)}dn$$

and

$$W(\phi) = \int_{N(E)\backslash N(\mathbb{A}_E)} \phi(n)\overline{\psi(n)}dn.$$

We define the *Bessel distribution* as

$$B'_{\pi'}(f') := \sum_i W'(\pi'(f')\phi'_i)\overline{W'(\phi'_i)}$$

and the *relative Bessel distribution* as

$$B_\pi(f) := \sum_j P(\pi(f)\phi_j)\overline{W(\phi_j)}$$

where the summations are over an orthonormal basis of $V_{\pi'}$ and V_π respectively. Flicker [Fli91], following related work of Jacquet-Lai [JL85] and Ye [Ye89], showed that for "matching functions" $f' \in C_c^\infty(U(2,\mathbb{A}_F))$ and $f \in C_c^\infty(\mathrm{GL}(2,\mathbb{A}_E))$, if π' maps to π under the nonstandard base change, then

$$\sum_i W'(\pi'(f')\phi'_i)\overline{W'(\phi'_i)} = \sum_j P(\pi(f)\phi_j)\overline{W(\phi_j)}.$$

In particular, this equality characterizes the image of the nonstandard base change lift associating every automorphic representation of $U(2,\mathbb{A}_F)$ to an automorphic representation of $\mathrm{GL}(2,\mathbb{A}_E)$ in terms of $\mathrm{GL}(2,\mathbb{A}_F)$ distinguished representations. The equality above is proved via the relative trace formula [Jac05], which tells us that for f and f' matching functions:

$$\int_{(N(F)\backslash N(\mathbb{A}_F))^2} K_{f'}(n_1,n_2)\psi'(n_1^{-1}n_2)dn_1dn_2 = \int_{\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A}_F)}\int_{N(E)\backslash N(\mathbb{A}_E)} K_f(h,n)\psi(n)dndh$$

where

$$K_f(x,y) = \sum_{\delta\in\mathrm{GL}(2,E)} f(x^{-1}\delta y).$$

The distributions $B'_{\pi'}(f')$ and $B_\pi(f)$ occur in the spectral expansions of the respective trace formulas.

In a different direction, Arthur developed a local version of the classical Arthur-Selberg trace formula ([Art89] and [Art91]). Let G be a connected reductive algebraic group over a local field F of characteristic zero. Diagonally embed $G(F)$ into $G(F)\times G(F)$. Then $L^2(G(F))$ is isomorphic to $L^2(G(F)\backslash G(F)\times G(F))$ by

$$\phi \mapsto ((y_1,y_2)\mapsto \phi(y_1^{-1}y_2)).$$

For $\phi \in L^2(G(F))$, let $(\rho(g_1,g_2)\phi)(x) = \phi(g_1^{-1}xg_2)$. The right regular representation of $G(F)\times G(F)$ on $L^2(G(F)\backslash G(F)\times G(F))$ is equivalent to ρ of $G(F)\times G(F)$ on $L^2(G(F))$. Thus to develop the local trace formula we look at $\rho(f)$ where $f = f_1\otimes f_2 \in C_c^\infty(G(F)\times G(F))$. Then

$$(\rho(f)\phi)(x) = \int_{G(F)}\int_{G(F)} f_1(g)f_2(y)\phi(g^{-1}xy)dgdy$$

is an integral operator on $L^2(G(F))$ with kernel

$$K_f(x,y) = \int_{G(F)} f_1(g)f_2(x^{-1}gy)dg.$$

The local trace formula develops an explicit formula for the regularized trace of $\rho(f)$.

The main result of this paper is that, when evaluated with matching functions, the two local trace formulas described in Theorems 1.3 and 1.4 below, i.e. the local Kuznetsov trace formula and the local relative trace formula, are equal. Thus there is an equality between their local distributions on the spectral sides. This equality is stated in Theorem 1.1. This is the natural local counterpart to Jacquet's global comparison. In order to develop the local relative trace formula stated in Theorem 1.4 we have to define a local regularized period integral, prove it is a $\mathrm{GL}(2,F)\times\mathrm{GL}(2,F)$-invariant linear functional and relate it to the truncated period integral that initially appears in the relative trace formula. We state these properties about the local regularized period integral in Proposition 1.2.

To describe this more precisely we need to introduce some further notation. Let E/F now denote an unramified extension of local non-Archimedean fields of characteristic 0. Let $\mathcal{O}_F$ (respectively $\mathcal{O}_E$) denote the ring of integers in F (respectively E). Let $H=\mathrm{GL}(2)/F$, $G=\mathrm{Res}_{E/F}H$ and let

$$G'=\mathrm{U}(2,F)=\left\{g\in G: {}^t\,\overline{g}\begin{pmatrix}0&1\\-1&0\end{pmatrix}g=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\}.$$

Let N' (resp. N) be the upper triangular unipotent matrices of G' (resp G) and let M' (resp. M) be the diagonal subgroup of G' (respectively G). Let Z (respectively Z') denote the center of G (respectively G'). For any subgroup X of G let $\widetilde{X}=Z\cap X\backslash X$ and let $X_H=X\cap H$. Let ψ' be an additive character on F with conductor $\mathcal{O}_F$ and let $\psi(x)=\psi'\circ\mathrm{tr}_{E/F}$. Let $f=f_1\otimes f_2\in C_c^\infty(\widetilde{G}(F)\times\widetilde{G}(F))$ and $f'=f_1'\otimes f_2'\in C_c^\infty(\widetilde{G}'(F)\times\widetilde{G}'(F))$.

We define the local Kuznetsov trace formula as the equality between the geometric expansion (in terms of orbital integrals) and spectral expansion (in terms of representations) of

$$\lim_{t\to\infty}\int_{(N'\times N')(F)}K_{f'}(n_1,n_2)\psi'(n_1^{-1}n_2)u(n_1,t)u(n_2,t),dn_1dn_2$$

and the local relative trace formula as the equality between the expansions of

$$\lim_{t\to\infty}\int_{\widetilde{H}(F)}\int_{N(F)}K_f(h,n)\psi(n)u(h,t)u(n,t)dndh.$$

In this local setting

$$K_f(x,y)=\int_{\widetilde{G}(F)}f_1(g)f_2(x^{-1}gy)dg,$$

$$K_{f'}(x,y)=\int_{\widetilde{G}'(F)}f_1'(g)f_2'(x^{-1}gy)dg$$

and $u(n,t)$ and $u(h,t)$ are truncation parameters defined analogously to Arthur's truncation [Art91] that are needed due to convergence issues.

We use the following ideas in this paper to rewrite these local trace formulas in terms of orbital integrals and representations:

- methods of Arthur from the local trace formula [Art91],
- methods of Jacquet and Ye from the relative trace formula ([Jac05], [Ye89]),
- Harish-Chandra's Plancherel formula ([HC84], [Wal03])
- Jacquet-Lapid-Rogawski's methods for regularizing period integrals [JLR99].

The power of the two trace formulas lies in the comparison. For "matching functions" the geometric expansions of the two local relative trace formulas are equal. By comparing the spectral expansions in these two trace formulas, we get an analogue of Jacquet's result, giving an identity between local Bessel distributions for functions on $U(2)$ and local relative Bessel distributions for functions on $\mathrm{GL}(2,E)$, and therefore local periods and local Whittaker functionals:

Theorem 1.1. *If σ is the nonstandard base change lift of the supercuspidal representation σ', and f' and f are matching functions, then*

$$d(\sigma')\sum_{S'\in\mathcal{B}(\sigma')}W'_{\sigma'}(\sigma'(f_2')S'\sigma'(f_1'^\vee))\overline{W'_{\sigma'}(S')}=d(\sigma)\sum_{S\in\mathcal{B}(\sigma)}P_\sigma(\sigma(f_2)S\sigma(f_1^\vee))\overline{W_\sigma(S)},$$

where $d(\sigma)$ is the formal degree of σ, $\mathcal{B}(\sigma)$ is an orthonormal basis of the Hilbert space of Hilbert-Schmidt operators on V_σ,

$$W'_{\sigma'}(S') = \int_{N'(F)} \mathrm{tr}(\sigma'(n)S')\psi'(n^{-1})dn,$$

$$W_\sigma(S) = \int_{N(F)} \mathrm{tr}(\sigma(n)S)\psi(n^{-1})dn,$$

and

$$P_\sigma(S) = \int_{\widetilde{H}(F)} \mathrm{tr}(\sigma(h)S)dh.$$

The Bessel and relative Bessel distributions factor locally into $B_{\pi_v}(f_v)$, but it is not clear how to normalize the local distributions. Each distribution above is the product of two local distributions and the equality above can be restated as

$$d(\sigma')B'_{\sigma'}(f'_2)B'_{\sigma'}(f'_1) = d(\sigma)B_{\sigma^*}(f_1)B_\sigma(f_2)$$

where B'_σ (B_σ) is a local (relative) Bessel distribution.

We note that $P_\sigma(S)$ is not a convergent integral if σ is not a discrete series representation. To develop the local relative trace formula we have to define a local regularized period integral. Let $K = G(\mathcal{O}_F)$ and let $P = NM$. For a principal series representation π of $\widetilde{G}$ and $u, v \in \pi$ we define the matrix coefficient $f_{u,v}(g) = \langle \pi(g)u, v\rangle$. Asymptotically on M, $f_{u,v}$ will equal a finite sum of functions of the form $e^{\lambda H_M(m)}$. We define the regularized period integral as:

$$\begin{aligned} P_\pi(f_{u,v}) &:= \int^*_{\widetilde{H}(F)} f_{u,v}(h)dh \\ &:= \int_{\widetilde{H}(F)} f_{u,v}(h)u(h,t)dh \\ &+ \int_{(\widetilde{K}_H\times\widetilde{K}_H)(F)} \int^\sharp_{\widetilde{M}_H^+(F)} D_{P_H}(m)f_{u,v}(k_1mk_2)(1-u(m,t))dmdk_1dk_2 \end{aligned}$$

where

$$\int^\sharp_{\widetilde{M}_H^+(F)} e^{\lambda H_M(m)}(1-u(m,t))dm$$

is the meromorphic continuation at $\nu = 0$ of

$$\int_{\widetilde{M}_H^+(F)} e^{(\nu+\lambda)H_M(m)}(1-u(m,t))dm$$

which is absolutely convergent for $\mathrm{Re}(\nu) \ll 0$.

We prove that with this definition the period integral is a $\mathrm{GL}(2,F)\times\mathrm{GL}(2,F)$-invariant linear functional, and we related it to the truncated period integral that initially appears in the relative trace formula as follows. By abuse of notation we identify a character χ of $\widetilde{M}(F)$ with a character χ of $E^\times$ by letting $\chi\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix} = \chi(a)\chi^{-1}(b)$. For $\lambda \in \mathbb{C}$ we let $\chi_\lambda(m) = \chi(m)e^{\lambda(H_P(m))}$. We let $I_P(\chi_\lambda)$ be the parabolically induced normalized representation acting on the Hilbert space $\mathcal{H}_P(\chi)$. Then for $S \in \mathcal{B}_P(\chi)$,

$$\mathrm{tr}(I_P(\chi_\lambda, k_1gk_2)S) = E_P(g, \Psi_S, \lambda)_{k_1,k_2},$$

where $E_P(g, \Psi, \lambda)$ is the Eisenstein integral and

$$(C^PE_P)(m,\psi,\lambda) = (c_{P|P}(1,\lambda)\psi)(m)e^{\lambda H_M(m)} + (c_{P|P}(w,\lambda)\psi)(m)e^{-\lambda H_M(m)}.$$

We fix a uniformizer ϖ in F (and E).

Proposition 1.2. *Fix a character χ of $E^\times$ such that $\chi(\varpi)=1$. Then for $t \gg 0$,*

$$\begin{aligned}
&\int_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda,h)S)u(h,t)dh \\
&= \int^*_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda,h)S)dh \\
&-\delta(\chi)(1+q^{-1})\left(\frac{q^{2\lambda(t+1)}}{1-q^{2\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H} c(1,\lambda)\Psi_S(1)_{k_1,k_2}dk_1dk_2 + \frac{q^{-2\lambda(t+1)}}{1-q^{-2\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H} c(w,\lambda)\Psi_S(1)_{k_1,k_2}dk_1dk_2\right),
\end{aligned}$$

where $\delta(\chi)=1$ if $\chi|_{\mathcal{O}_F^\times}=1$ and $\delta(\chi)=0$ if $\chi|_{\mathcal{O}_F^\times}\neq 1$.

Let $|\cdot|_E$ denote the normalized valuation on E. Denote the action of the nontrivial element in $\mathrm{Gal}(E/F)$ on $x\in E$ by $\bar{x}$. Denote by $N_{E/F}$ the norm map from $E^\times$ to $F^\times$. Let $E^1=\{x\in E^\times : N_{E/F}(x)=1\}$. Let η denote an element in $G(F)$ such that $\overline{\eta}^{-1}\eta=\begin{pmatrix}0&1\\1&0\end{pmatrix}$.

We define

$$D'_{\chi'_\lambda}(f')=\sum_{S'\in\mathcal{B}_P(\chi')} W'_{\chi'_\lambda}(S'_\lambda[f'])\overline{W'_{\chi'_\lambda}(S')},$$

and

$$D_{\chi_\lambda}(f)=\sum_{S\in\mathcal{B}_P(\chi)} P_{\chi_\lambda}(S_\lambda[f])\overline{W_{\chi_\lambda}(S)}$$

where

$$W'_{\chi'_\lambda}(S')=\lim_{t\to\infty}\int_{N'(F)}\operatorname{tr}(I_{P'}(\chi'_\lambda,n)S')\psi'(n^{-1})u(n,t)dn,$$

$$W_{\chi_\lambda}(S)=\lim_{t\to\infty}\int_{N(F)}\operatorname{tr}(I_P(\chi_\lambda,n)S)\psi(n^{-1})u(n,t)dn$$

and

$$S_\lambda[f]=d(\chi)I_P(\chi_\lambda,f_2)SI_P(\chi_\lambda,f_1^\vee).$$

We let $\Pi_2(\widetilde{G}'(F))$ be a set of equivalence classes of irreducible, tempered square integrable representations of $\widetilde{G}'(F)$. We identify unitary characters on $\widetilde{M}'(F)$ with characters on $E^\times$ that are trivial on E^1. We let $\{\Pi_2(\widetilde{M}'(F))\}$ be a set of representatives of unitary characters χ' on $\widetilde{M}'(F)$ such that $\chi'(\varpi)=1$. We let $\mu(\chi'_\lambda)$ be Harish-Chandra's μ-function. We take the analogous definitions for $\widetilde{G}(F)$.

Theorem 1.3 (Local Kuznetsov trace formula)**.** *For any $f'=f'_1\otimes f'_2\in C_c^\infty(\widetilde{G}'(F)\times\widetilde{G}'(F))$,*

$$\begin{aligned}
&\lim_{t\to\infty}\int_{N'(F)}\int_{N'(F)} K_{f'}(n_1,n_2)\psi'(n_1^{-1}n_2)u(n_1,t)u(n_2,t)dn_1dn_2 \\
&=\int_{a\in E^\times/E^1} O'\left(f_1,\psi',a\right)O'\left(f'_2,\bar{\psi}',a\right)|a|_E d^\times a \\
&=\sum_{\sigma'\in\Pi_2(\widetilde{G}'(F))} d(\sigma')D'_{\sigma'}(f')+\frac{1}{2}\sum_{\chi'\in\{\Pi_2(\widetilde{M}'(F))\}} d(\chi')\int_0^{\frac{\pi i}{\log q}}\mu(\chi'_\lambda)D'_{\chi'_\lambda}(f')d\lambda
\end{aligned}$$

where

$$O'(f',\psi',a)=\int_{N'(F)}\int_{N'(F)} f'\left(n_1^{-1}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}a&0\\0&\overline{a}^{-1}\end{pmatrix}n_2\right)\overline{\psi'(n_1n_2^{-1})}dn_1dn_2.$$

Theorem 1.4 (Local relative trace formula)**.** *For any $f = f_1 \otimes f_2 \in C_c^\infty(\widetilde{G}(F) \times \widetilde{G}(F))$,*

$$\begin{aligned}
&\lim_{t\to\infty} \int_{\widetilde{H}(F)} \int_{N(F)} K_f(h,n)\psi(n)u(h,t)u(n,t)dndh \\
&= \int_{a\in E^\times/E^1} O\left(f_1,\psi,a\right) O\left(f_2,\bar{\psi},a\right) |a|_E d^\times a \\
&= \sum_{\sigma\in\Pi_2(\widetilde{G}(F))} d(\sigma)D_\sigma(f) + \frac{1}{2} \sum_{\substack{\chi\in\{\Pi_2(\widetilde{M}(F))\} \\ \chi^2\neq 1, \chi|_{F^\times}=1}} \widetilde{D}_\chi(f) \\
&+\frac{1}{2} \sum_{\substack{\chi\in\{\Pi_2(\widetilde{M}(F))\} \\ \chi|_{E^1}=1}} d(\chi) \int_0^{\frac{\pi i}{\log q}} \mu(\chi_\lambda) D_{\chi_\lambda}(f)d\lambda
\end{aligned}$$

where

$$O(f,\psi,a) = \int_{\widetilde{H}(F)} \int_{N(F)} f\left(h^{-1}\eta \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} n\right) \psi(n)dndh.$$

The representations that occur on the right hand side of Theorem 1.4 are exactly the representations that are in the image of the nonstandard base change lift on $G'(F)$. The extra discrete term $\widetilde{D}_\chi(f)$ corresponds to the representations that lift from the discrete series on $\widetilde{G}'(F)$ to the principal series on $\widetilde{G}(F)$.

In addition to the spectral comparison, these local trace formulas also have applications on the geometric side. If we define the inner product of two functions g_1, g_2 on $E^\times/E^1$ by

$$\langle g_1, g_2\rangle = \int_{a\in E^\times/E^1} |a|_E g_1(a)g_2(a)d^\times a,$$

then

Proposition 1.5 (Orthogonality Relations)**.** *For f_1 (resp. f_1') and f_2 (resp. f_2') matrix coefficients of the supercuspidal representations σ_1 (resp. σ_1')and σ_2 (resp. σ_2'),*

$$\langle O'\left(f_1',\psi',\cdot\right), O'\left(f_2',\psi'^{-1},\cdot\right)\rangle \neq 0 \text{ iff } \sigma_1' \sim \sigma_2'$$

and

$$\langle O\left(f_1,\psi,\cdot\right), O\left(f_2,\psi^{-1},\cdot\right)\rangle \neq 0 \text{ iff } \sigma_1 \sim \sigma_2.$$

Using Macdonald's formula, we can explicitly evaluate the local trace formulas for spherical functions, a special subset of smooth functions. In this case there is no discrete spectrum and the spectral sides each reduce to a single term.

Example 1.6. *For any bi-K' invariant function $f \in C_c^\infty(\widetilde{G}'(F) \times \widetilde{G}'(F))$ and $t \gg 0$,*

$$\begin{aligned}
&\lim_{t\to\infty} \int_{(N'(F))^2} K_{f'}(n_1,n_2)\psi'(n_1^{-1}n_2)u(n_1,t)u(n_2,t)dn_1dn_2 \\
&= \int_0^{\frac{\pi i}{\log q}} m(\lambda)\hat{T}_{f'}(\lambda) \left| \int_{N'(F)} \phi'_\lambda(n)\overline{\psi'(n)}u(n,t)dn \right|^2 d\lambda \\
&= \int_0^{\frac{\pi i}{\log q}} \hat{T}_{f'}(\lambda)(1-q^{2\lambda})(1-q^{-2\lambda})(1+q^{-2}-q^{-1+2\lambda}-q^{-1-2\lambda})d\lambda.
\end{aligned}$$

For any bi-K invariant function $f \in C_c^\infty(\widetilde{G}(F) \times \widetilde{G}(F))$ and $t \gg 0$,

$$\begin{aligned}
&\lim_{t\to\infty} \int_{\widetilde{H}(F)} \int_{N(F)} K_f(h,n)\psi(n)u(h,t)u(n,t)dndh \\
&= \int_0^{\frac{\pi i}{\log q}} m(\lambda)\hat{T}_f(\lambda) \left(\int_{\widetilde{H}(F)}^{*} \phi_\lambda(h)dh \right) \left(\overline{\int_{N(F)} \phi_\lambda(n)\overline{\psi(n)}u(n,t)dn} \right) d\lambda
\end{aligned}$$

$$= \int_0^{\frac{\pi i}{\log q}} \hat{T}_f(\lambda)(1+q^{2\lambda})(1+q^{-2\lambda})(1+q^{-2}+q^{-1+2\lambda}+q^{-1-2\lambda})d\lambda$$

where

$$\int^*_{\widetilde{H}(F)} \phi_\lambda(h)dh = \int_{\widetilde{H}(F)} \phi_\lambda(h)u(h,t)dh - \frac{1+q^{-1}}{1+q^{-2}}\left(c(\lambda)\frac{q^{2\lambda(t+1)}}{1-q^{2\lambda}} + c(-\lambda)\frac{q^{-2\lambda(t+1)}}{1-q^{-2\lambda}}\right),$$

$m(\lambda)$ is the Plancherel measure, ϕ_λ is a spherical matrix coefficient corresponding to the characteristic function of K, $I_P(\chi_\lambda, f_2)I_P(\chi_\lambda, f_1^\vee)\phi_\lambda = \hat{T}(\lambda)\phi_\lambda$ and

$$c(\lambda) = \frac{1-q^{-2-4\lambda}}{1-q^{-4\lambda}}.$$

Thus when f is the image of f' under the base change map between their Hecke algebras, $\hat{T}'(\lambda)$ and $\hat{T}(\lambda)$ are Laurent polynomials in $q^{2\lambda}$ that are equal under the mapping $q^{2\lambda} \to -q^{2\lambda}$. As expected, when f is the image of f' under the base change map between their Hecke algebras, the trace formulas are equal.

The rest of this paper is organized as follows. In Section 2 we define notation and give normalizations of measures. In Section 3 we develop the local Kuznetsov trace formula. For the geometric expansion we rewrite our trace formula in terms of orbital integrals corresponding to the $N'\backslash G'/N'$ cosets. The orbital integrals for these cosets depend on the truncation and are intertwined. It is only through the multiplication of the two orbital integrals together, integration over the space of double cosets, and the non triviality of the character ψ', that we are able to untangle the orbital integral for f_1' from the orbital integral for f_2'. For the spectral expansions we apply Harish-Chandra's Plancherel formula to rewrite the local kernel in terms of representations. We are left with integrals of matrix coefficients against the character over the unipotent subgroup. By the smoothness of the matrix coefficients and the appearance of the character, we show these distributions stabilize for t large.

In Section 4 we develop the local relative trace formula of $H\backslash G/N$. In the spectral expansion we have truncated integrals of matrix coefficients over H that do not converge without the truncation. We define the regularized period integral $P_\sigma(S)$. We use the asymptotics of matrix coefficients of tempered representations to prove the truncated integral is a polynomial exponential function in the truncation parameter t. We define the regularized integral as the constant term of this polynomial, and prove that this is the relevant term in the local relative trace formula and an H invariant linear functional.

In Section 5 we compare our two local trace formulas. There is a bijection between the admissible $N'\backslash G'/N'$ cosets and the admissible $H\backslash G/N$ cosets and both of these sets can be parameterized by $E^\times/E^1$. This allows us to compare the geometric sides. By work of Ye, we know that for any f' there is an f such that the orbital integrals are equal for corresponding cosets. Thus, by their geometric expansions, our local trace formulas are equal. This tells us the spectral expansions are equal and we get the equality of local distributions.

This paper would not have come into being had it not been for my teacher and advisor, Jonathan Rogawski. These thoughts originated as my PhD thesis under his direction, and his ideas, support, and guidance were critical to its completion. I am fortunate and will be forever grateful to have had him as a mentor. He could explain complicated math in a clear and simple way that aimed at the heart of the problem. He served, and continues to serve, as the role model of the inquisitive, patient, and approachable mathematician.

2. Notation

Let F be a non-Archimedean local field of characteristic 0 and odd residual characteristic q. Let E be an unramified quadratic extension of F. Let $\mathcal{O}_F$ (respectively $\mathcal{O}_E$) denote the ring of integers in F (respectively E). Let ϖ denote a uniformizer in the maximal ideal of $\mathcal{O}_F$. Thus ϖ is also a uniformizer in E. Let $v(\cdot)$ denote the valuation on F, extended to E. Let $|\cdot|_F$ (respectively $|\cdot|_E$) denote the normalized valuation on F (respectively E). Thus for $a \in F^\times$, $|a|_E = |a|_F^2$. Denote the action of the nontrivial element in $\mathrm{Gal}(E/F)$ on $x \in E$ by $\bar{x}$. Denote by $N_{E/F}$ the norm map from $E^\times$ to $F^\times$.

Let $H = \mathrm{GL}(2)/F$ and let $G = \mathrm{Res}_{E/F} H$, the restriction of scalars of GL(2) from E to F. Thus $G(F) = \mathrm{GL}(2, E)$. Let

$$G' = \mathrm{U}(2, F) = \left\{ g \in G : {}^t\, \overline{g} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

We note that by defining the quasi-split unitary group in this way $SL(2, F) \subset G'(F)$. Let N' (resp. N) be the upper triangular unipotent matrices of G' (resp. G). Let M' (respectively M) be the diagonal subgroup of G' (respectively G). That is,

$$M'(F) = \left\{ \begin{pmatrix} a & 0 \\ 0 & \overline{a}^{-1} \end{pmatrix} : a \in E^\times \right\}$$

and

$$M(F) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in E^\times \right\}.$$

Occasionally by abuse of notation we let $n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $a = \begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix}$. Let $P = NM$ and $P' = N'M'$. Let $K = G(\mathcal{O}_F)$ and $K' = G'(\mathcal{O}_F)$. Let Z (respectively Z') denote the center of G (respectively G'). For any subgroup X of G' let $\widetilde{X} = Z' \cap X \backslash X$ and $X_H = X \cap H$. By abuse of notation we identify a character χ of $\widetilde{M}(F)$ with a character χ of $E^\times$ by letting $\chi \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \chi(a)\chi^{-1}(b)$.

Let ψ' be an additive character on F with conductor $\mathcal{O}_F$. Let ψ be the additive character on E defined by $\psi(x) = \psi'(x + \overline{x})$. By abuse of notation we will also denote by ψ and ψ' the corresponding characters on $N(F)$ and $N'(F)$ respectively. Let $f = f_1 \otimes f_2 \in C_c^\infty(\widetilde{G}(F) \times \widetilde{G}(F))$ and $f' = f_1' \otimes f_2' \in C_c^\infty(\widetilde{G}'(F) \times \widetilde{G}'(F))$. For a function f on G, let $f^\vee(g) = f(g^{-1})$.

To define the local Kuznetsov trace formula and local relative trace formula we first insert Arthur's local truncation [Art91, §3], and then take the limit of the integral of the truncated function. For Arthur's truncation we multiply our function by the characteristic function of a large compact subset of $\widetilde{G}(F)$. For $g \in \widetilde{G}(F)$, $t \in \mathbb{Z}^+$, let

$$u(g, t) = \begin{cases} 1 & \text{if } g = zk_1 \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} k_2, \text{ for some } k_1, k_2 \in K, z \in Z(F), 0 \le v(\alpha) \le t \\ 0 & \text{otherwise.} \end{cases}$$

We note that

$$u\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, t \right) = \begin{cases} 1 & \text{if } x \in \varpi^{[\frac{-t}{2}]} \mathcal{O}_E \\ 0 & \text{otherwise} \end{cases},$$

where $[x]$ is the integral part of x.

If X is a closed subgroup of $\widetilde{G}(F)$ with the subgroup topology, $\mathrm{supp}(u(\cdot, t)) \cap X$ is a compact set.

We normalize the Haar measure dx on F so that $\mathrm{vol}(\mathcal{O}_F) = 1$. We define the multiplicative measure $d^\times x$ on $F^\times$ as $d^\times x = \frac{1}{1-q^{-1}} \frac{1}{|x|_F} dx$. Thus $\mathrm{vol}(\mathcal{O}_F^\times) = 1$. We let $N(F)$ and $M(F)$ have the measures induced by dx and $d^\times x$. We normalize the Haar measure dk on K so that $\mathrm{vol}(K) = 1$. We define the measure dg on $G(F)$ by

$$\int_{G(F)} f(g) dg = \int_{M(F)} \int_{N(F)} \int_K f(mnk) dk dn dm.$$

We define dg' on $G'(F)$ similarly. We normalize Haar measure on $\widetilde{K}$ by taking $\mathrm{vol}(\widetilde{K})$=1.

We let $d^\times a$ be the unique Haar measure on $E^\times / E^1$ such that $\mathrm{vol}(\mathcal{O}_E^\times / E^1) = \frac{1}{1+q^{-1}}$.

3. The local Kuznetsov trace formula for U(2)

In this section we develop a local Kuznetsov trace formula for the quasi-split unitary group in two variables. We expand this local Kuznetsov trace formula geometrically in terms of separate orbital integrals for f_1' and f_2'. Then we use Harish-Chandra's Plancherel formula to rewrite this expression spectrally in terms of representations.

We define the local Kuznetsov trace formula for $f' = f_1' \otimes f_2' \in C_c^\infty(\widetilde{G}'(F) \times \widetilde{G}'(F))$ as the equality between the geometric and spectral expansions of

$$\lim_{t\to\infty} \int_{(N'\times N')(F)} K_{f'}(n_1, n_2)\psi(n_1^{-1}n_2)u(n_1,t)u(n_2,t)dn_1dn_2$$

where

$$K_{f'}(n_1,n_2) = \int_{\widetilde{G}'(F)} f_1'(g)f_2'(n_1^{-1}gn_2)dg.$$

We show that for a fixed f' this limit stabilizes, that is, there exists a T such that for all $t' \geq T$,

$$\begin{aligned}&\int_{(N'\times N')(F)} K_{f'}(n_1, n_2)\psi'(n_1^{-1}n_2)u(n_1,t')u(n_2,t')dn_1dn_2\\ =& \lim_{t\to\infty} \int_{(N'\times N')(F)} K_{f'}(n_1, n_2)\psi(n_1^{-1}n_2)u(n_1,t)u(n_2,t)dn_1dn_2.\end{aligned}$$

3.1. **The geometric expansion.** We rewrite

$$\lim_{t\to\infty} \int_{N'(F)}\int_{N'(F)} K_{f'}(n_1, n_2)\psi'(n_1^{-1}n_2)u(n_1,t)u(n_2,t)dn_1dn_2$$

as an integral over admissible cosets of a product of an orbital integral for f_1' and an orbital integral for f_2'.

3.1.1. *Integration formula.* Let $w = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$. For $a \in E^\times$, let $\beta_a = \begin{pmatrix} a & 0 \\ 0 & \overline{a}^{-1}\end{pmatrix}$ and $\gamma_a = w\begin{pmatrix} a & 0 \\ 0 & \overline{a}^{-1}\end{pmatrix}$. Let $E^1 = \{a \in E^\times : N_{E/F}(a) = 1\}$. By the Bruhat decomposition, $G' = P' \sqcup P'wP'$. Thus

$$\{\beta_a : a \text{ is in a set of representatives for } E^\times/E^1\} \bigcup \{\gamma_a : a \text{ is in a set of representatives for } E^\times/E^1\}$$

is a set of representatives for the double cosets of $N'(F)\backslash\widetilde{G}'(F)/N'(F)$.

For $g \in \widetilde{G}'$ let

$$C_g(N'\times N') = \{(n_1,n_2) \in N'\times N' : n_1^{-1}gn_2 = zg \text{ for some } z \in Z'\}.$$

Definition 3.1. *An element $g \in \widetilde{G}'(F)$ and its corresponding orbit is called **admissible** if the map*

$$C_g(N'(F)\times N'(F)) \to \mathbb{C} : (n_1,n_2) \mapsto \psi'(n_1^{-1}n_2)$$

*is trivial. An orbit which is not admissible is called **inadmissible**.*

By a simple calculation we see that

$$\begin{aligned}&C_{\beta_a}(N'(F)\times N'(F)) = \left\{\left(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & \frac{x}{a\overline{a}} \\ 0 & 1\end{pmatrix}\right) : x \in F\right\} \text{ and}\\ &C_{\gamma_a}(N'(F)\times N'(F)) = 1.\end{aligned}$$

Thus the orbits represented by $\{\beta_a : a \in (E^\times - E^1)/E^1\}$ are inadmissible and the orbits represented by $\{\beta_1\} \cup \{\gamma_a : a \in E^\times/E^1\}$ are admissible.

We use the following integration formula to rewrite $K_{f'}(n_1,n_2)$ as an integral over only the admissible cosets. Unlike in the global case the trivial admissible coset, β_1, will not contribute to the local Kuznetsov trace formula.

For any $F \in C_c(\widetilde{G}'(F))$,

$$\int_{\widetilde{G}'(F)} F(g)dg = \int_{E^\times/E^1}\int_{(N'\times N')(F)} F(n_1^{-1}\gamma_a n_2)dn_1dn_2|a|_E d^\times a. \tag{3.1}$$

3.1.2. *Separating the orbital integrals.* Let

$$K^t(f') = \int_{N'(F)} \int_{N'(F)} K_{f'}(n_1, n_2)\psi(n_1^{-1}n_2)u(n_1, t)u(n_2, t)dn_1dn_2.$$

Clearly $K^t(f')$ is absolutely convergent because f_1' and $u(\cdot, t)$ have compact support on $\widetilde{G}'(F)$ and $N'(F)$ respectively. By changing the order of integration and using (3.1) we see that $K^t(f')$ equals

$$\int_{E^\times/E^1} \int_{(N'\times N')(F)} \int_{(N'\times N')(F)} f_1'(\hat{n}_1^{-1}\gamma_a\hat{n}_2)f_2'(n_1^{-1}\hat{n}_1^{-1}\gamma_a\hat{n}_2n_2) \times\psi'(n_1^{-1}n_2)u(n_1, t)u(n_2, t)dn_1dn_2d\hat{n}_1d\hat{n}_2|a|_Ed^\times a.$$

We note that the above integral is absolutely convergent as the map from $N'(F)\times E^\times/E^1 \times N'(F)$ to $\widetilde{G}'(F)$ defined by $(n_1, a, n_2) \mapsto n_1^{-1}\gamma_an_2$ is injective and f_1' has compact support. By a change of variables we have

$$K^t(f') = \int_{E^\times/E^1} K^t(\gamma_a, f')|a|_Ed^\times a,$$

where

$$K^t(\gamma_a, f') = \int_{(N'\times N')(F)} \int_{(N'\times N')(F)} f_1'(\hat{n}_1^{-1}\gamma_a\hat{n}_2)f_2'(n_1^{-1}\gamma_an_2)\psi'(n_1^{-1}\hat{n}_1\hat{n}_2^{-1}n_2) \times u(\hat{n}_1^{-1}n_1, t)u(\hat{n}_2^{-1}n_2, t)dn_1dn_2d\hat{n}_1d\hat{n}_2.$$

To complete the geometric expansion of the local Kuznetsov trace formula we rewrite $K^t(f')$ for $t \gg 0$ as an integral of two separate orbital integrals. We begin by examining the dependence of the integrand on the truncation.

Lemma 3.2. *Let $f_1', f_2' \in C_c(\widetilde{G}'(F))$. For each $t_0 > 0$ there exists a $T > 0$ such that for all $t \geq T$,*

$$f_1'(x_1^{-1}\gamma y_1)f_2'(x_2^{-1}\gamma y_2)u(x_1^{-1}x_2, t)u(y_1^{-1}y_2, t_0) = f_1'(x_1^{-1}\gamma y_1)f_2'(x_2^{-1}\gamma y_2)u(y_1^{-1}y_2, t_0)$$

for all $x_1, x_2, y_1, y_2, \gamma \in \widetilde{G}'(F)$.

Proof. Let $\Omega_1 = \mathrm{supp}(f_1'), \Omega_2 = \mathrm{supp}(f_2')$ and $\Omega_3 = \mathrm{supp}(u(\cdot, t_0)) \cap \widetilde{G}'(F)$. These sets are all compact on $\widetilde{G}'(F)$. If $f_1'(x_1^{-1}\gamma y_1)f_2'(x_2^{-1}\gamma y_2)u(y_1^{-1}y_2, t_0) \neq 0$, then the following conditions must hold:

- $x_1^{-1} \in \Omega_1y_1^{-1}\gamma^{-1}$;
- $x_2 \in \gamma y_2\Omega_2^{-1}$;
- $y_1^{-1}y_2 \in \Omega_3$.

Thus if $f_1'(x_1^{-1}\gamma y_1)f_2'(x_2^{-1}\gamma y_2)u(y_1^{-1}y_2, t_0) \neq 0$, then $x_1^{-1}x_2 \in \Omega_1\Omega_3\Omega_2^{-1}$. Because this is a compact set, there exists a $T > 0$ such that $\Omega_1\Omega_3\Omega_2^{-1} \subseteq \mathrm{supp}(u(g, T))$. The lemma now follows. □

Now we use this lemma, along with the character ψ, to separate the two orbital integrals. Let K_1 be an open compact subgroup of $\widetilde{G}'(F)$ such that f_1' and f_2' are bi-K_1-invariant. There exists a positive constant c such that $\begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix} \in K_1$ for all $a \in 1 + \varpi^c\mathcal{O}_E$. By abuse of notation, in the proof of the following lemma we let

$$\varpi^n = \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^{-n} \end{pmatrix} \text{ and } a = \begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix}.$$

Lemma 3.3. *For $f' = f_1' \otimes f_2' \in C_c^\infty(\widetilde{G}'(F) \times \widetilde{G}'(F))$, there exists a T such that for all $t \geq T$ and $n \in \mathbb{Z}'$,*

$$\int_{a\in\varpi^n\mathcal{O}_E^\times/E^1} K^t(\gamma_a, f')d^\times a = \int_{a\in\varpi^n\mathcal{O}_E^\times/E^1} O'(f_1', \psi', a)O'(f_2', \bar{\psi}', a)d^\times a, \tag{3.2}$$

where

$$O'(f', \psi', a) = \int_{N'(F)} \int_{N'(F)} f'(n_1^{-1}\gamma_an_2)\overline{\psi'(n_1^{-1}n_2)}dn_1dn_2.$$

Proof. We show that there is a hidden truncation on the right hand side of (3.2) that comes from the fact that the two orbital integrals are simultaneously evaluated at the same γ_a. By definition

$$\begin{aligned}
&\int_{a\in\varpi^n\mathcal{O}_E^\times/E^1} K^t(\gamma_a, f')d^\times a \\
=&\int_{a\in\mathcal{O}_E^\times/E^1}\int_{(N'\times N')(F)} f_1'(\hat{n}_1^{-1}w\varpi^n a\hat{n}_2)\psi'(\hat{n}_1\hat{n}_2^{-1}) \\
\times&\int_{(N'\times N')(F)} f_2'(n_1^{-1}w\varpi^n an_2)\psi(n_1^{-1}n_2)u(\hat{n}_1^{-1}n_1,t)u(\hat{n}_2^{-1}n_2,t)dn_2dn_1d\hat{n}_2d\hat{n}_1d^\times a \\
=&\sum_{\eta\in\mathcal{O}_E^\times/(1+\varpi^c\mathcal{O}_E)E^1}\int_{a\in(1+\varpi^c\mathcal{O}_E)E^1/E^1}\int_{(N'\times N')(F)} f_1'(\hat{n}_1^{-1}w\varpi^n\eta a\hat{n}_2)\psi'(\hat{n}_1\hat{n}_2^{-1}) \\
&\times\int_{(N'\times N')(F)} f_2'(n_1^{-1}w\varpi^n\eta an_2)\psi'(n_1^{-1}n_2)u(\hat{n}_1^{-1}n_1,t)u(\hat{n}_2^{-1}n_2,t)dn_2dn_1d\hat{n}_2d\hat{n}_1d^\times a.
\end{aligned}$$

By a change of variables and the fact that f' is locally constant the above line equals

$$\begin{aligned}
&\sum_{\eta\in\mathcal{O}_E^\times/(1+\varpi^c\mathcal{O}_E)E^1}\int_{(N'\times N')(F)} f_1'(\hat{n}_1^{-1}w\varpi^n\eta\hat{n}_2)\psi'(\hat{n}_1) \\
\times&\int_{(N'\times N')(F)} f_2'(n_1^{-1}w\varpi^n\eta n_2)\psi'(n_1^{-1})u(\hat{n}_1^{-1}n_1,t)dn_1d\hat{n}_1 \\
\times&\int_{a\in(1+\varpi^c\mathcal{O}_E)E^1/E^1}\psi'(a^{-1}\hat{n}_2^{-1}n_2a)u(a^{-1}\hat{n}_2^{-1}n_2a,t)d^\times adn_2d\hat{n}_2.
\end{aligned}$$

We can rewrite the inner integral as

$$\begin{aligned}
&u(\hat{n}_2^{-1}n_2,t)\int_{a\in(1+\varpi^c\mathcal{O}_E)E^1/E^1}\psi'\left((n_2-\hat{n}_2)(a\bar{a})^{-1}\right)d^\times a \\
=&u(\hat{n}_2^{-1}n_2,t)\int_{b\in1+\varpi^c\mathcal{O}_F}\psi'(b(n_2-\hat{n}_2))d^\times b \\
=&u(\hat{n}_2^{-1}n_2,t)\frac{1}{1-q^{-1}}\psi'(n_2-\hat{n}_2)\int_{b\in\varpi^c\mathcal{O}_F}\psi'(b(n_2-\hat{n}_2))db \\
=&u(\hat{n}_2^{-1}n_2,t)u(\hat{n}_2^{-1}n_2,2c)\frac{q^{-c}}{1-q^{-1}}\psi'(\hat{n}_2^{-1}n_2).
\end{aligned}$$

Thus for $t\geq 2c$,

$$\begin{aligned}
&\int_{a\in\varpi^n\mathcal{O}_E^\times/E^1} K^t(\gamma_a, f')d^\times a \\
=&\int_{a\in\varpi^n\mathcal{O}_E^\times/E^1}\int_{(N'\times N')(F)} f_1'(\hat{n}_1^{-1}\gamma_a\hat{n}_2)\psi'(\hat{n}_1\hat{n}_2^{-1}) \\
\times&\int_{(N'\times N')(F)} f_2'(n_1^{-1}\gamma_a n_2)\psi'(n_1^{-1}n_2)u(\hat{n}_1^{-1}n_1,t)u(\hat{n}_2^{-1}n_2,2c)dn_2dn_1d\hat{n}_2d\hat{n}_1d^\times a.
\end{aligned}$$

By Lemma 3.2 there exists a $T > 0$ such that for all $t \geq \max\{T, 2c\}$

$$\begin{aligned}
&\int_{a\in\varpi^n\mathcal{O}_E^\times/E^1} K^t(\gamma_a, f')d^\times a \\
&= \int_{a\in\varpi^n\mathcal{O}_E^\times/E^1}\int_{(N'\times N')(F)} f_1'\left(\hat{n}_1^{-1}\gamma_a\hat{n}_2\right)\psi'(\hat{n}_1\hat{n}_2^{-1})d\hat{n}_1 \\
&\times \int_{(N'\times N')(F)} f_2'\left(n_1^{-1}\gamma_a n_2\right)\psi'(n_1^{-1}n_2)u(\hat{n}_2^{-1}n_2, 2c)dn_2dn_1d\hat{n}_2d^\times a \\
&= \int_{a\in\varpi^n\mathcal{O}_E^\times/E^1}\int_{(N'\times N')(F)} f_1'\left(\hat{n}_1^{-1}\gamma_a\hat{n}_2\right)\psi'(\hat{n}_1\hat{n}_2^{-1})d\hat{n}_2d\hat{n}_1 \\
&\times \int_{(N'\times N')(F)} f_2'\left(n_1^{-1}\gamma_a n_2\right)\psi'(n_1^{-1}n_2)dn_2dn_1d^\times a.
\end{aligned}$$

□

We have shown that the truncated local Kuznetsov trace formula stabilizes.

Proposition 3.4. *For any $f' = f_1'\otimes f_2' \in C_c^\infty(\widetilde{G}'(F)\times\widetilde{G}'(F))$ and $t \gg 0$,*

$$\begin{aligned}
&\int_{N'(F)}\int_{N'(F)} K_{f'}(n_1,n_2)\psi'(n_1^{-1}n_2)u(n_1,t)u(n_2,t)dn_1dn_2 \\
&= \int_{a\in E^\times/E^1} O'\left(f_1',\psi',a\right)O'\left(f_2',\bar{\psi}',a\right)|a|_E d^\times a.
\end{aligned}$$

3.2. **The spectral expansion.** Now we derive a spectral expansion for the local Kuznetsov trace formula,

$$\lim_{t\to\infty}\int_{N'(F)}\int_{N'(F)} K_{f'}(n_1,n_2)\psi'(n_1^{-1}n_2)u(n_1,t)u(n_2,t)dn_1dn_2.$$

Our main tool is the Plancherel formula for p-adic groups which was first stated, with an outlined proof, by Harish-Chandra [HC84]. Silberger later filled in an important proof of one of the steps in the theorem [Sil96]. More recently Waldspurger provided a complete proof [Wal03].

As Arthur does in [Art91, §2] we begin by rewriting $K_{f'}(x,y)$ using the Plancherel formula. First we introduce some additional notation. For an irreducible representation (σ, V_σ) of $G'(F)$ let $\mathcal{B}(\sigma)$ be the Hilbert space of Hilbert-Schmidt operators on V_σ. The inner product on $\mathcal{B}(\sigma)$ is defined as

$$\langle S, S'\rangle := \operatorname{tr}(SS'^*)$$

for $S, S' \in \mathcal{B}(\sigma)$ where $\operatorname{tr}(SS'^*) = \sum_{\text{o.n.b.}V_\sigma}\langle SS'^*u_i, u_i\rangle$ and this sum converges absolutely and does not depend on the basis. For a discrete series representation σ of a group G let $d(\sigma)$ be the formal degree of σ.

Let $\Pi_2(\widetilde{G}'(F))$ be a set of representatives for the equivalence classes of irreducible, tempered square integrable representations of $\widetilde{G}'(F)$ and let $\{\Pi_2(\widetilde{M}'(F))\}$ be a set of representatives of unitary characters χ on $\widetilde{M}'(F)$ such that χ is a character on $\mathcal{O}_E$ that is trivial on E^1. For a character χ of $M'(F)$ and $\lambda \in \mathbb{C}$, let $\chi_\lambda(m) = \chi(m)e^{\lambda(H_{P'}(m))}$. For $\chi \in \Pi_2(M'(F))$, $I_{P'}^{G'}(\chi) = I_{P'}(\chi)$ is the normalized induced representation of $G(F)$ which acts on a Hilbert space $\mathcal{H}_{P'}(\chi)$ of vector valued functions on K'. Let $\mathcal{B}_{P'}(\chi)$ be a fixed K'-finite orthonormal basis of the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}_{P'}(\chi)$.

Let $m(\sigma)$ be the Plancherel density. We normalize our measures following [Art91, §1]. The Plancherel density satisfies $m(\chi_\lambda) = d(\chi)\mu(\chi_\lambda)$, where $\mu(\chi_\lambda)$ is Harish-Chandra's μ-function.

For a fixed $x \in G'(F)$, let

$$h(v) = \int_{\widetilde{G}'(F)} f_1'(xu)f_2'(uvx)du.$$

Then $h \in C_c^\infty(\widetilde{G}'(F))$ and $K_{f'}(x,y) = h(yx^{-1})$ so by the Plancherel formula,

$$K_{f'}(x,y) = \sum_{\sigma\in\Pi_2(\widetilde{G}'(F))} d(\sigma)\operatorname{tr}(\sigma(R(yx^{-1})h)) + \frac{1}{2}\sum_{\chi\in\{\Pi_2(\widetilde{M}'(F))\}}\int_0^{\frac{\pi i}{\log q}} \operatorname{tr}(I_{P'}(\chi, R(yx^{-1})h))m(\chi_\lambda)d\lambda.$$

Because

$$I_{P'}(\chi_\lambda, R(yx^{-1})h) = I_{P'}(\chi_\lambda, f_1'^{\vee}) I_{P'}(\chi_\lambda, x) I_{P'}(\chi_\lambda, f_2')(I_{P'}(\chi_\lambda, y))^*,$$

we have

$$\begin{aligned}
\mathrm{tr}(I_{P'}(\chi_\lambda, R(yx^{-1})h)) &= \sum_{S \in \mathcal{B}_{P'}(\chi)} (I_{P'}(\chi_\lambda, f_1'^{\vee}) I_{P'}(\chi_\lambda, x) I_{P'}(\chi_\lambda, f_2'), S^*)\overline{(I_{P'}(\chi_\lambda, y), S^*)} \\
&= \sum_{S \in \mathcal{B}_{P'}(\chi)} \mathrm{tr}(I_{P'}(\chi_\lambda, f_1'^{\vee}) I_{P'}(\chi_\lambda, x) I_{P'}(\chi_\lambda, f_2') S)\overline{\mathrm{tr}(I_{P'}(\chi_\lambda, y) S)} \\
&= \sum_{S \in \mathcal{B}_{P'}(\chi)} \mathrm{tr}(I_{P'}(\chi_\lambda, x) S_{\chi_\lambda}[f'])\overline{\mathrm{tr}(I_{P'}(\chi_\lambda, y) S)},
\end{aligned}$$

where $S_{\chi_\lambda}[f'] = I_{P'}(\chi_\lambda, f_2') S I_{P'}(\chi_\lambda, f_1'^{\vee})$.

For $f' \in C_c^\infty(\widetilde{G}'(F))$, π an admissible representation, $\pi(f')$ has finite rank. Thus the sum over S is a finite sum of an orthonormal basis of operators on $\mathcal{H}_P(\chi)^{K_0}$ for some open compact K_0.

Putting everything together we have

$$\begin{aligned}
&\int_{N'(F)} \int_{N'(F)} K_{f_1' \otimes f_2'}(n_1, n_2) \psi'(n_1^{-1} n_2) u(n_1, t) u(n_2, t) dn_1 dn_2 \\
&= \sum_{\sigma \in \Pi_2(\widetilde{G}'(F))} d(\sigma) \sum_{S \in \mathcal{B}(\sigma)} \int_{N'(F)} \mathrm{tr}(\sigma(n)(\sigma(f_2') S \sigma(f_1'^{\vee}))) \psi'(n^{-1}) u(n, t) dn \overline{\int_{N'(F)} \mathrm{tr}(\sigma(n) S) \psi'(n^{-1}) u(n, t) dn} \\
&+ \frac{1}{2} \sum_{\chi \in \{\Pi_2(\widetilde{M}'(F))\}} d(\chi) \\
&\times \int_0^{\frac{\pi i}{\log q}} \left(\sum_{S \in \mathcal{B}_{P'}(\chi)} \int_{N'(F)} \mathrm{tr}(I_{P'}(\chi_\lambda, n) S_{\chi_\lambda}[f']) \psi'(n^{-1}) u(n, t) dn \overline{\int_{N'(F)} \mathrm{tr}(I_{P'}(\chi_\lambda, n) S) \psi'(n^{-1}) u(n, t) dn} \right) \mu(\chi_\lambda) d\lambda.
\end{aligned}$$

To finish the spectral expansion we show that the above integrals stabilize. We first note that in the discrete series case the above integrals are absolutely convergent without any truncation for reasons similar to those in Section 4.2.1.

Lemma 3.5 (Spectral Stabilization). *For any complex valued function ϕ on $\widetilde{G}'(F)$ that is bi-invariant under an open compact subgroup, there exists a positive integer c such that for all $t \geq c$,*

$$\int_{N'(F)} \phi(n) \psi'(n) u(n, t) dn = \int_{N'(F)} \phi(n) \psi'(n) u(n, c) dn.$$

Note: This c only depends on the open compact subgroup under which ϕ is bi-invariant.

Proof. Let K_1 be an open compact subgroup of $\widetilde{G}'(F)$ under which ϕ is bi-invariant. K_1 must contain a neighborhood of the identity, so there exists a positive integer c' such that for $a, b, c, d \in \varpi^{c'} \mathcal{O}_F$, $\begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix} \in K_1$. We show that for $m > c'$,

$$\int_{\varpi^{-m} \mathcal{O}_F^\times} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \psi'(x) dx = 0.$$

We note that

$$\begin{pmatrix} x' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x'^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x'x \\ 0 & 1 \end{pmatrix}.$$

Thus for $x' \in 1 + \varpi^{c'} \mathcal{O}_F$,

$$\phi\left(\begin{pmatrix} 1 & x'x \\ 0 & 1 \end{pmatrix}\right) = \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right).$$

Hence

$$\int_{\varpi^{-m}\mathcal{O}_F^\times} \phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\psi'(x)dx$$
$$= \sum_{\alpha\in\mathcal{O}_F^\times/(1+\varpi^{c'}\mathcal{O}_F)} \int_{\varpi^{-m}(1+\varpi^{c'}\mathcal{O}_F)} \phi\left(\begin{pmatrix}1 & \alpha x\\ 0 & 1\end{pmatrix}\right)\psi'(\alpha x)dx$$
$$= \sum_{\alpha\in\mathcal{O}_F^\times/(1+\varpi^{c'}\mathcal{O}_F)} \phi\left(\begin{pmatrix}1 & \varpi^{-m}\alpha\\ 0 & 1\end{pmatrix}\right)\psi'(\varpi^{-m}\alpha)\int_{\varpi^{c'-m}\mathcal{O}_F}\psi'(x)dx.$$

The last line equals 0 for $m > c'$. Thus for $t > 2c'$,

$$\int_{N'(F)} \phi(n)\psi'(n)u(n,t)dn$$
$$= \int_{N'(F)} \phi(n)\psi'(n)u(n,2c')dn.$$

□

We have now proved the following.

Proposition 3.6. *For any $f' = f_1'\otimes f_2' \in C_c^\infty(\widetilde{G}'(F)\times\widetilde{G}'(F))$,*

$$\lim_{t\to\infty}\int_{N'(F)}\int_{N'(F)} K_{f'}(n_1,n_2)\psi'(n_1^{-1}n_2)u(n_1,t)u(n_2,t)dn_1dn_2$$
$$= \sum_{\sigma'\in\Pi_2(\widetilde{G}'(F))} d(\sigma')D'_{\sigma'}(f') + \frac{1}{2}\sum_{\chi'\in\{\Pi_2(\widetilde{M}'(F))\}} d(\chi')\int_0^{\frac{\pi i}{\log q}} D'_{\chi'_\lambda}(f')\mu(\chi'_\lambda)d\lambda,$$

where

$$D'_{\sigma'}(f') = \sum_{S\in\mathcal{B}(\sigma')} W'_{\sigma'}(\sigma'(f_2')S\sigma'(f_1'^\vee))\overline{W'_{\sigma'}(S)},$$

$$W'_{\sigma'}(S) = \int_{N'(F)} \mathrm{tr}(\sigma'(n)S)\psi'(n^{-1})dn,$$

$$D'_{\chi'_\lambda}(f') = \sum_{S\in\mathcal{B}_{P'}(\chi')} W'_{\chi'_\lambda}(I_{P'}(\chi'_\lambda,f_2')SI_{P'}(\chi'_\lambda,f_1'^\vee))\overline{W'_{\chi'_\lambda}(S)}$$

and

$$W'_{\chi'_\lambda}(S) = \lim_{t\to\infty}\int_{N'(F)} \mathrm{tr}\left(I_{P'}(\chi'_\lambda,n)S\right)\psi'(n^{-1})u(n,t)dn.$$

We note that Theorem 1.3 now follows from the results of Propositions 3.4 and 3.6.

4. The local relative trace formula and periods for PGL(2)

In this section we define a local relative trace formula for PGL(2). We expand this local relative trace formula geometrically in terms of separate orbital integrals of f_1 and f_2. Then we use Harish-Chandra's Plancherel formula to rewrite this expression spectrally in terms of representations. We define a regularized period integral, show that it is an H-invariant linear functional and that it is the term that appears in the spectral expansion of the local relative trace formula.

We define the local relative trace formula for $f = f_1\otimes f_2 \in C_c^\infty(\widetilde{G}(F)\times\widetilde{G}(F))$ as the equality between the geometric and spectral expansions of

$$\lim_{t\to\infty}\int_{\widetilde{H}(F)}\int_{N(F)} K_f(h,n)\psi(n)u(h,t)u(n,t)dndh$$

where

$$K_f(h,n) = \int_{\widetilde{G}(F)} f_1(g)f_2(h^{-1}gn)dg.$$

As with the local Kuznetsov trace formula, we show that for a fixed f this limit stabilizes.

4.1. **The geometric expansion.** We will rewrite

$$\lim_{t\to\infty}\int_{\widetilde{H}(F)}\int_{N(F)}K_f(h,n)\psi(n)u(h,t)u(n,t)dndh$$

as an integral over admissible cosets of a product of an orbital integral for f_1 and an orbital integral f_2.

4.1.1. *Integration formula.* As pointed out in [JLR99, §VI.13], by [Spr85],

$$G(F)=H(F)P(F)\sqcup H(F)\eta P(F),$$

where η is any element in $G(F)$ such that $\overline{\eta}^{-1}\eta=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Let $\eta_a=\eta\begin{pmatrix}a&0\\0&1\end{pmatrix}$ and $\gamma_\alpha=\begin{pmatrix}1&0\\0&\alpha+\sqrt{\tau}\end{pmatrix}$, where $E=F(\sqrt{\tau})$. Then

$$\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}\cup\{\gamma_\alpha:\alpha\in F\}\cup\left\{\eta_a: a\text{ is in a set of representatives for }E^\times/E^1\right\}$$

is a set of representatives for the double cosets of $\widetilde{H}(F)\backslash\widetilde{G}(F)/N(F)$.

For $g\in\widetilde{G}$, let

$$C_g(\widetilde{H}\times N)=\{(h,n)\in\widetilde{H}\times N: h^{-1}gn=zg\text{ for some }z\in Z\}.$$

Definition 4.1. *An element $g\in\widetilde{G}(F)$ and its corresponding orbit is called **admissible** if the map*

$$C_g(\widetilde{H}(F)\times N(F))\to\mathbb{C}:(h,n)\mapsto\psi(n)$$

*is trivial. An orbit which is not admissible is called **inadmissible**.*

By a short calculation we see that

$$C_{\gamma_\alpha}(\widetilde{H}(F)\times N(F))=\left\{\left(\begin{pmatrix}1&y\\0&1\end{pmatrix},\begin{pmatrix}1&y(\alpha+\sqrt{\tau})\\0&1\end{pmatrix}\right):y\in F\right\}\text{ and}$$
$$C_{\eta_a}(\widetilde{H}(F)\times N(F))=1.$$

Thus the orbits represented by $\{\gamma_\alpha:\alpha\in F^\times\}\cup\{1\}$ are inadmissible and the orbits represented by $\{\eta_a: a\in E^\times/E^1\}\cup\{\gamma_0\}$ are admissible.

We have the following integration formula. For any $F\in C_c(\widetilde{G}(F))$,

$$\int_{\widetilde{G}(F)}F(g)dg=\int_{E^\times/E^1}\int_{\widetilde{H}(F)\times N(F)}F(h^{-1}\eta_a n)dndh|a|_E d^\times a. \tag{4.1}$$

4.1.2. *Separating the orbital integrals.* Let

$$R^t(f)=\int_{\widetilde{H}(F)}\int_{N(F)}K_f(h,n)\psi(n)u(h,t)u(n,t)dndh.$$

$R^t(f)$ is absolutely convergent because $f_1(g),u(h,t)$ and $u(n,t)$ have compact support on $\widetilde{G}(F)$, $\widetilde{H}(F)$ and $N(F)$ respectively. By changing the order of integration and applying (4.1) we see that $R^t(f)$ equals

$$\int_{E^\times/E^1}\int_{\widetilde{H}(F)\times N(F)}\int_{\widetilde{H}(F)\times N(F)}f_1(h_1^{-1}\eta_a n_1)f_2(h_2^{-1}h_1^{-1}\eta_a n_1 n_2)$$
$$\times\psi(n_2)u(h_2,t)u(n_2,t)dn_2dh_2dn_1dh_1|a|_E d^\times a.$$

We note that the above integral is absolutely convergent as the map from $\widetilde{H}(F)\times E^\times/E^1\times N(F)$ to $\widetilde{G}(F)$ defined by $(h,a,n)\mapsto h^{-1}\eta_a n$ is injective and f_1 has compact support. By a change of variables we have

$$R^t(f)=\int_{E^\times/E^1}R^t(\eta_a,f)|a|_E d^\times a,$$

where

$$R^t(\eta_a, f) = \int_{\widetilde{H}(F)\times N(F)} \int_{\widetilde{H}(F)\times N(F)} f_1(h_1^{-1}\eta_a n_1) f_2(h_2^{-1}\eta_a n_2)\psi(n_1^{-1}n_2) \times u(h_1^{-1}h_2, t)u(n_1^{-1}n_2, t)dn_2 dh_2 dn_1 dh_1.$$

To complete the geometric expansion of the local relative trace formula we rewrite $R^t(f)$ for $t \gg 0$ as an integral of a product of two separate orbital integrals that are not truncated.

Let K_1 be an open compact subgroup of $\widetilde{G}(F)$ such that f_1 and f_2 are bi-K_1-invariant. There exists a $c' > 0$ such that $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in K_1$ for $a \in 1 + \varpi^{c'}\mathcal{O}_E$. By abuse of notation we let

$$\varpi^n = \begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix} \text{ and } a = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 4.2. *For $f \in C_c^\times(\widetilde{G}(F) \times \widetilde{G}(F))$, there exists a $T > 0$ such that for all $t \geq T$,*

$$\int_{\varpi^n \mathcal{O}_E^\times / E^1} R^t(\eta_a, f) d^\times a = \int_{\varpi^n \mathcal{O}_E^\times / E^1} O'(f_1', \psi, a) O(f_2, \bar{\psi}, a) d^\times a$$

for all $n \in \mathbb{Z}$ where

$$O(f, \psi, a) = \int_{\widetilde{H}(F)} \int_{N(F)} f(h^{-1}\eta_a n)\overline{\psi(n)} dn dh.$$

Proof. This proof is very similar to the proof of Lemma 3.3 so we will omit some details.

$$\begin{aligned}
&\int_{\varpi^n \mathcal{O}_E^\times / E^1} R^t(\eta_a, f) d^\times a \\
&= \int_{a\in\mathcal{O}_E^\times / E^1} \int_{\widetilde{H}(F)} \int_{N(F)} f_1(h_1^{-1}\eta\varpi^n a n_1)\psi(n_1^{-1}) \\
&\times \int_{\widetilde{H}(F)} \int_{N(F)} f_2(h_2^{-1}\eta\varpi^n a n_2)\psi(n_2) u(h_1^{-1}h_2, t)u(n_1^{-1}n_2, t) dn_2 dh_2 dn_1 dh_1 d^\times a \\
&= \sum_{\beta \in \mathcal{O}_E^\times/(1+\varpi^{c'}\mathcal{O}_E)} \int_{a\in(1+\varpi^{c'}\mathcal{O}_E)/E^1} \int_{\widetilde{H}(F)} \int_{N(F)} f_1(h_1^{-1}\eta\varpi^n \beta a n_1)\psi(n_1^{-1}) \\
&\times \int_{\widetilde{H}(F)} \int_{N(F)} f_2(h_2^{-1}\eta\varpi^n \beta a n_2)\psi(n_2) u(h_1^{-1}h_2, t)u(n_1^{-1}n_2, t) dn_2 dh_2 dn_1 dh_1 d^\times a.
\end{aligned}$$

By a change of variables this equals

$$\begin{aligned}
&\sum_{\beta \in \mathcal{O}_E^\times/(1+\varpi^{c'}\mathcal{O}_E)E^1} \int_{\widetilde{H}(F)} \int_{N(F)} f_1(h_1^{-1}\eta\varpi^n\beta n_1) \int_{\widetilde{H}(F)} \int_{N(F)} f_2(h_2^{-1}\eta\varpi^n\beta n_2) u(h_1^{-1}h_2, t) dh_2 dh_1 \\
&\times \int_{a\in(1+\varpi^{c'}\mathcal{O}_E)E^1/E^1} \psi(a^{-1}n_1^{-1}n_2 a) u(a^{-1}n_1^{-1}n_2 a, t) d^\times a dn_2 dn_1 \\
&= \int_{a\in\varpi^n\mathcal{O}_E^\times/E^1} \int_{\widetilde{H}(F)} \int_{N(F)} f_1(h_1^{-1}\eta_a n_1)\psi(n_1^{-1}) \\
&\times \int_{\widetilde{H}(F)} \int_{N(F)} f_2(h_2^{-1}\eta_a n_2)\psi(n_2) u(h_1^{-1}h_2, t) u(n_1^{-1}n_2, c') dn_2 dh_2 dn_1 dh_1 d^\times a
\end{aligned}$$

for $t \geq c'$. By Lemma 3.2 (whose proof is identical if we replace $\widetilde{G}'(F)$ with $\widetilde{G}(F)$), there exists a $T > 0$ such that for all $t \geq \max\{T, c'\}$

$$\begin{aligned}
&\int_{a\in\varpi^n\mathcal{O}_E^\times/E^1} R^t(\eta_a, f)d^\times a \\
&= \int_{a\in\varpi^n\mathcal{O}_E^\times/E^1}\int_{\widetilde{H}(F)}\int_{N(F)} f_1(h_1^{-1}\eta_a n_1)\psi(n_1^{-1})dh_1 \\
&\times \int_{\widetilde{H}(F)}\int_{N(F)} f_2(h_2^{-1}\eta_a n_2)\psi(n_2)u(n_1^{-1}n_2, c')dn_2dh_2dn_1d^\times a \\
&= \int_{a\in\varpi^n\mathcal{O}_E^\times/E^1}\int_{\widetilde{H}(F)}\int_{N(F)} f_1(h_1^{-1}\eta_a n_1)\psi(n_1^{-1})dn_1dh_1 \\
&\times \int_{\widetilde{H}(F)}\int_{N(F)} f_2(h_2^{-1}\eta_a n_2)\psi(n_2)dn_2dh_2d^\times a.
\end{aligned}$$

□

We have proved the following proposition.

Proposition 4.3. *For any $f = f_1 \otimes f_2 \in C_c^\infty(\widetilde{G}(F)\times\widetilde{G}(F))$,*

$$\begin{aligned}
&\lim_{t\to\infty}\int_{\widetilde{H}(F)}\int_{N(F)} K(h,n)\psi(n)u(h,t)u(n,t)dndh \\
&= \int_{a\in E^\times/E^1} O\left(f_1,\psi,a\right)O\left(f_2,\overline{\psi},a\right)|a|_E d^\times a.
\end{aligned}$$

Here, as in the local Kuznetsov trace formula, we have actually shown that the limit of the truncated local relative trace formula stabilizes.

4.2. **The spectral expansion and period integrals.** We want to get a spectral expansion for the local relative trace formula,

$$\lim_{t\to\infty}\int_{\widetilde{H}(F)}\int_{N(F)} K_f(h,n)\psi(n)u(h,t)u(n,t)dndh.$$

As in the previous section, we expand the kernel via the Plancherel formula:

$$\begin{aligned}
&\int_{\widetilde{H}(F)}\int_{N(F)} K(h,n)\psi(n)u(h,t)u(n,t)dndh \\
&= \sum_{\sigma\in\Pi_2(\widetilde{G}(F))} d(\sigma)D_\sigma^t(f) + \frac{1}{2}\sum_{\chi\in\{\Pi_2(\widetilde{M}(F))\}} d(\chi)\int_0^{\frac{\pi i}{\log q}}\mu(\chi_\lambda)D_{\chi_\lambda}^t(f)d\lambda
\end{aligned}\tag{4.2}$$

where

$$D_\sigma^t(f) = \sum_{S\in\mathcal{B}(\sigma)} P_\sigma^t(\sigma(f_2)S\sigma(f_1^\vee))\overline{W_\sigma^t(S)},$$

$$D_{\chi_\lambda}^t(f) = \sum_{S\in\mathcal{B}_P(\chi)} P_{I_P(\chi_\lambda)}^t(I_P(\chi_\lambda, f_2)SI_P(\chi_\lambda, f_1^\vee))\overline{W_{I_P(\chi_\lambda)}^t(S)},$$

$$W_\pi^t(S) = \int_{N(F)} \mathrm{tr}(\pi(n)S)\psi(n^{-1})u(n,t)dn$$

and

$$P_\pi^t(S) = \int_{\widetilde{H}(F)} \mathrm{tr}(\pi(h)S)u(h,t)dh.$$

By Lemma 3.5, there exists a positive integer c_1, such that for $t > c_1$,

$$W_\pi^t(S) = \int_{N(F)} \mathrm{tr}(\pi(n)S)\psi(n^{-1})u(n,c_1)dn.$$

Thus as in the previous section, we define

$$W_\pi(S) = \lim_{t\to\infty} \int_{N(F)} \operatorname{tr}(\pi(n)S)\psi(n^{-1})u(n,t)dn.$$

To finish the spectral expansion of the local relative trace formula we need to define the regularized integral

$$\int^*_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda, h)S)dh$$

because $\operatorname{tr}(I_P(\chi_\lambda, -)S)$ is not integrable over $\widetilde{H}(F)$.

Many of the techniques in this section are inspired by the work of Jacquet-Lapid-Rogawski in [JLR99]. In that paper they define a regularized period integral for an automorphic form ϕ on $G(\mathbb{A})$ integrated over H where G is a reductive group over a number field F and H is the fixed point set of an involution θ of G. They focus on the case $G = \operatorname{Res}_{E/F} H$ where E/F is a quadratic extension and they obtain explicit results for $G = \mathrm{GL}(n, E)$, $H = \mathrm{GL}(n, F)$.

Let $\delta_P\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right) = \left|\frac{\alpha}{\beta}\right|_E$,

$$H_M\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right) = \begin{pmatrix}\log|\alpha|_E & 0\\ 0 & \log|\beta|_E\end{pmatrix}$$

and

$$\rho_P\left(H_M\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right)\right) = \frac{1}{2}\log|\frac{\alpha}{\beta}|_E.$$

We note that for $g \in H$, $H_M(g) = 2H_{M_H}(g)$. For $\lambda \in \mathbb{C}$, $e^{\lambda H_M(m)} = |\frac{\alpha}{\beta}|_E^\lambda$.

We recall the Cartan decomposition $H(F) = K_H M_H^+ K_H$ where $M_H^+ = \left\{\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} : v(\frac{\alpha}{\beta}) \le 0\right\}$. Then

$$\int_{\widetilde{H}(F)} f(h)dh = \int_{\widetilde{K}_H}\int_{\widetilde{K}_H}\int_{\widetilde{M_H}^+} D_{P_H}(m)f(k_1mk_2)dmdk_2dk_1$$

where

$$D_{P_H}\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right) = \begin{cases}|\frac{\alpha}{\beta}|_F(1+|\varpi|_F) & v(\alpha/\beta)\le 0\\ 0 & v(\alpha/\beta) > 0\end{cases}$$

for any absolutely integrable function f.

We note that

$$1 - u\left(\begin{pmatrix}1 & \\ & \alpha\end{pmatrix}, t\right) = \begin{cases}0, & 0 \le v(\alpha) \le t\\ 1, & v(\alpha) > t\end{cases}$$

Let $\mathcal{M}(g) \in M^+$ be such that $g = k_1\mathcal{M}(g)k_2$, $k_1, k_2 \in K$.

For Re $\nu <$ −Re λ,

$$\int_{\widetilde{M}_H^+} e^{(\nu+\lambda)H_M(m)}(1-u(m,t))dm = \sum_{n=t+1}^{\infty} q^{2n(\nu+\lambda)} = \frac{q^{(t+1)2(\nu+\lambda)}}{1-q^{2(\nu+\lambda)}}. \tag{4.3}$$

We write

$$\int^\sharp_{\widetilde{M}_H^+} e^{\lambda H_M(m)}(1-u(m,t))dm$$

to denote the meromorphic continuation at $\nu = 0$ of (4.3). This is well defined so long as $\lambda \ne 0$. If

$$\phi(k_1mk_2) = \sum_{i=1}^{r}\phi_i(k_1,k_2)f_i(m)e^{\lambda_i H(m)}, k_1, k_2 \in K_H, m = \begin{pmatrix}1 & \\ & \varpi^n\end{pmatrix}, n \ge 0, f_i \in C_c(\widetilde{M}(F)) \tag{4.4}$$

where $\lambda_i \neq -1/2$ we define for $t \gg 0$

$$\int^{\sharp}_{\widetilde{H}(F)} \phi(h)(1-u(h,t))dh = \sum_{i=1}^{r} \int_{\widetilde{K}_H \times \widetilde{K}_H} \phi_i(k_1,k_2) \int^{\sharp}_{\widetilde{M}_H^+} D_{P_H}(m) e^{\lambda_i H_M(m)}(1-u(m,t))dm$$
$$= (1+q^{-1}) \sum_{i=1}^{r} \int_{\widetilde{K}_H \times \widetilde{K}_H} \phi_i(k_1,k_2) \int^{\sharp}_{\widetilde{M}_H^+} e^{(\lambda_i+1/2)H_M(m)}(1-u(m,t))dm.$$

If ϕ is a matrix coefficient of $I_P(\chi_\lambda)$ where $\chi(\varpi)=1$ then by smoothness and the asymptotics of matrix coefficients there exists a function $C^P\phi$ of the form in (4.4) with $\lambda_i \in \{\lambda - 1/2, -\lambda-1/2\}$ and for $n \gg 0$, $C^P\phi\left(k_1 \begin{pmatrix} 1 & \\ & \varpi^n \end{pmatrix} k_2\right) = \phi\left(k_1 \begin{pmatrix} 1 & \\ & \varpi^n \end{pmatrix} k_2\right)$. Note that the condition for the regularized integral to exist is now that $\lambda \neq 0$.

Definition 4.4. *For any matrix coefficient ϕ of $I_P(\chi_\lambda)$ where $\chi(\varpi)=1$ and $\lambda \neq 0$,*

$$\int^{*}_{\widetilde{H}(F)} f(h)dh$$
$$:= \int_{\widetilde{H}(F)} f(h)u(h,t)dh + \int_{\widetilde{K}_H \times \widetilde{K}_H} \int^{\sharp}_{\widetilde{M}_H^+} D_{P_H}(m) f(k_1 m k_2)(1-u(m,t))dm dk_1 dk_2$$

for $t \gg 0$.

One can check that this definition of the regularized integral is independent of t and agrees with the usual integral if we start with something that is integrable. Now we will prove that it is $\widetilde{H}$-invariant and then we will explicitly relate the regularized period to the truncated period that occurs in the local trace formula.

Let $\phi^{h_0}(x) = \phi(xh_0)$ for $h_0 \in \widetilde{H}$. Note that if ϕ is a matrix coefficient of π then ϕ^{h_0} is as well.

Lemma 4.5. *For any matrix coefficient ϕ of $I_P(\chi_\lambda)$ where $\chi(\varpi)=1$ and $\lambda \neq 0$, $h_0 \in H$ and $t \gg 0$,*

$$\int_{\widetilde{K}_H \times \widetilde{K}_H} \int^{\sharp}_{\widetilde{M}_H} D_{P_H}(m) \phi^{h_0}(k_1 m k_2)(1-u(k_1 m k_2 h_0,t))dm dk_1 dk_2$$
$$= \int_{\widetilde{K}_H \times \widetilde{K}_H} \int^{\sharp}_{\widetilde{M}_H} D_{P_H}(m) \phi(k_1 m k_2)(1-u(m,t))dm dk_1 dk_2$$

Proof. For Re $\nu \ll 0$ and $t \gg 0$,

$$\int_{\widetilde{K}_H \times \widetilde{K}_H} \int_{\widetilde{M}_H} D_{P_H}(m) \phi^{h_0}(k_1 m k_2) e^{\nu(H_M(\mathcal{M}(k_1 m k_2 h_0)))}(1-u(k_1 m k_2 h_0,t))dm dk_1 dk_2$$
$$= \int_{\widetilde{H}(F)} \phi(hh_0) e^{\nu(H_M(\mathcal{M}(hh_0)))}(1-u(hh_0,t))dh$$
$$= \int_{\widetilde{H}(F)} \phi(h) e^{\nu(H_M(\mathcal{M}(h)))}(1-u(h,t))dh$$
$$= \int_{\widetilde{K}_H \times \widetilde{K}_H} \int_{\widetilde{M}_H} D_{P_H}(m) \phi(k_1 m k_2) e^{\nu(H_M(m))}(1-u(m,t))dm dk_1 dk_2$$

by the invariance of Haar measure, since both sides are absolutely convergent. For $t \gg 0$, if $\mathcal{M}(h) > t$ where $h = k_1 \mathcal{M}(h) k_2$, then $\mathcal{M}(hh_0) = \mathcal{M}(h)\mathcal{M}(k_2 h_0)$. Thus both sides of the equation above have a meromorphic continuation whose value at $\nu = 0$ gives the statement of the lemma. □

Proposition 4.6 (H-invariance)**.** *For ϕ a matrix coefficient of $I_P(\chi_\lambda)$ where $\chi(\varpi)=1$ and $\lambda \neq 0$, and $h_0 \in H$,*

$$\int^{*}_{\widetilde{H}(F)} \phi^{h_0}(h)dh = \int^{*}_{\widetilde{H}(F)} \phi(h)dh.$$

Proof. By the definition of the regularized integrals, the statement of the proposition will follow once we prove the following equality

$$\int_{\widetilde{K}_H\times\widetilde{K}_H}\int^{\sharp}_{\widetilde{M}_H^+} D_{P_H}(m)\phi^{h_0}(k_1mk_2)(1-u(m,t))dmdk_1dk_2-\int_{\widetilde{K}_H\times\widetilde{K}_H}\int^{\sharp}_{\widetilde{M}_H^+} D_{P_H}(m)\phi(k_1mk_2)(1-u(m,t))dmdk_1dk_2$$
$$=\int_{\widetilde{H}(F)}\phi(h)u(h,t)dh-\int_{\widetilde{H}(F)}\phi^{h_0}(h)u(h,t)dh.$$

First we note that by Lemma 4.5

$$\int_{\widetilde{K}_H\times\widetilde{K}_H}\int^{\sharp}_{\widetilde{M}_H^+} D_{P_H}(m)\phi^{h_0}(k_1mk_2)(1-u(m,t))dmdk_1dk_2-\int_{\widetilde{K}_H\times\widetilde{K}_H}\int^{\sharp}_{\widetilde{M}_H^+} D_{P_H}(m)\phi(k_1mk_2)(1-u(m,t))dmdk_1dk_2$$
$$=\int_{\widetilde{K}_H\times\widetilde{K}_H}\int^{\sharp}_{\widetilde{M}_H^+} D_{P_H}(m)\phi(k_1mk_2)(1-u(mk_2h_0^{-1},t)))dmdk_1dk_2-\int_{\widetilde{K}_H\times\widetilde{K}_H}\int^{\sharp}_{\widetilde{M}_H^+} D_{P_H}(m)\phi(k_1mk_2)(1-u(m,t))dmdk_1dk_2.$$

For fixed h_0 and t sufficiently large, $u(\cdot h_0^{-1},t)-u(\cdot,t)$ has support contained in an annulus. From this fact one can easily check that the previous line is equal to the convergent integral

$$\begin{aligned}&\int_{\widetilde{K}_H\times\widetilde{K}_H}\int_{\widetilde{M}_H(F)^+} D_{P_H}(m)\phi(k_1mk_2)[u(m,t)-u(mk_2h_0^{-1},t)]dmdk_1dk_2.\\ =&\int_{\widetilde{H}(F)}\phi(h)[u(h,t)-u(hh_0^{-1},t)]dh\\ =&\int_{\widetilde{H}(F)}\phi(h)u(h,t)dh-\int_{\widetilde{H}(F)}\phi^{h_0}(h)u(h,t)dh.\end{aligned}$$

□

To derive an explicit formula in terms of regularized periods for the truncated periods of matrix coefficients that appear in the trace formula we recall some definitions of Harish-Chandra's. For σ an admissible, tempered representation of G, $\mathcal{A}_\sigma(G)$ is the space of functions on G spanned by K-finite matrix coefficients of σ, $\mathcal{A}_{\text{temp}}(G)$ is the sum of $\mathcal{A}_\sigma(G)$ over all admissible tempered representations of G and $\mathcal{A}_2(G)$ is the sum of $\mathcal{A}_\sigma(G)$ over all unitary, square integrable representations. For τ a finite dimensional, unitary, two-sided representation of K,

$$\mathcal{A}_\sigma(G,\tau)=\{f\in\mathcal{A}_\sigma(G)\otimes V_\tau : f(k_1gk_2)=\tau(k_1)f(g)\tau(k_2), g\in G, k_1,k_2\in K\}.$$

Then $\mathcal{A}_{\text{temp}}(G,\tau)$ and $\mathcal{A}_2(G,\tau)$ are defined similarly.

Let $\tau_M=\tau|_{K\cap M}$. By [HC84, §3] for $f\in\mathcal{A}_\sigma(G,\tau)$ there exists a unique function $C^Pf\in\mathcal{A}(M,\tau_M)$ such that

$$\lim_{|\frac{\alpha}{\beta}|_E\to\infty}\left|\delta_P\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right)^{1/2} f\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right)-(C^Pf)\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right)\right|=0.$$

We call C^Pf the weak constant term of f.

For two parabolics P_1,P_2 with Levi component M, let

$$V_{P_1|P_2}=\{v\in V : \tau(n_1)v\tau(n_2)=v, n_1\in N_{P_1}\cap K, n_2\in N_{P_2}\cap K\}$$

and let $\tau_{P_1|P_2}$ be the subrepresentation of τ_M on $V_{P_1|P_2}$. For $\Psi\in\mathcal{A}_2(M,\tau_{P|P})$ and $\lambda\in[0,\frac{\pi i}{\log q}]$, the Eisenstein integral $E_P(g,\Psi,\lambda)\in\mathcal{A}_{\text{temp}}(G,\tau)$ is defined as

$$E_P(g,\Psi,\lambda)=\int_K\tau(k)^{-1}\Psi_P(kg)e^{(\lambda+\rho_P)(H_P(kg))}dk$$

where Ψ_P extends Ψ to G by

$$\Psi_P(nmk)=\Psi(m)\tau(k)\text{ for }n\in N(E), m\in M, k\in K.$$

The weak constant term of the Eisenstein integral uniquely defines Harish-Chandra's c-functions [HC84, §6]. For each element w in the Weyl group W of $\widetilde{G}$, the c-function $c_{P|P}(w,\lambda)$ is a linear map from $\mathcal{A}_2(M,\tau_{P|P})$ to $\mathcal{A}_2(M,\tau_{P|\overline{P}})$ such that

$$\begin{aligned}&(C^P E_P)(m,\Psi,\lambda)\\ =&(c_{P|P}(1,\lambda)\Psi)(m)e^{\lambda H_M(m)}+(c_{P|P}(w,\lambda)\Psi)(m)e^{-\lambda H_M(m)}\end{aligned}$$

where w is a representative for the nontrivial element in the Weyl group of $\widetilde{G}$. Let $c_{P|P}(s,\lambda)_\chi$ denote the restriction of $c_{P|P}(s,\lambda)$ to $\mathcal{A}_\chi(M,(\tau_\Gamma)_{P|P})$. We have

$$\mu(\chi_\lambda)^{-1}=c_{P|P}(s,\lambda)^*_\chi c_{P|P}(s,\lambda)_\chi.$$

For the rest of this section we let $c(1,\lambda)=c_{P|P}(1,\lambda)_\chi$ and $c(w,\lambda)=c_{P|P}(w,\lambda)_\chi$.

We note that the S we consider are actually in $\mathcal{H}_P(\chi)^{K_0}$ for some open compact K_0. Harish-Chandra [HC76, §7] gives an isomorphism $S\to\Psi_S$ from $\mathrm{End}(\mathcal{H}_P(\chi)^K)$ onto $\mathcal{A}_\chi(M,(\tau)_{P|P})$ where V_τ is a particular subspace of $L^2(K\times K)$ such that

$$\mathrm{tr}(I_P(\chi_\lambda,k_1gk_2)S)=E_P(g,\Psi_S,\lambda)_{k_1,k_2}.$$

We can now relate the definition of the regularized integral to what appears in the local relative trace formula.

Proposition 4.7. *For* $\chi=(\chi,\chi^{-1})\in\{\Pi_2(\widetilde{M}(F)\}$, $\lambda\neq 0$, $t\gg 0$,

$$\begin{aligned}&\int^*_{\widetilde{H}(F)}\mathrm{tr}(I_P(\chi_\lambda,h)S)dh\\ =&\int_{\widetilde{H}(F)}\mathrm{tr}(I_P(\chi_\lambda,h)S)u(h,t)dh+\\ &\delta(\chi)(1+q^{-1})\left(\frac{q^{2\lambda(t+1)}}{1-q^{2\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H}c(1,\lambda)\Psi_S(1)_{k_1,k_2}dk_1dk_2+\frac{q^{-2\lambda(t+1)}}{1-q^{-2\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H}c(w,\lambda)\Psi_S(1)_{k_1,k_2}dk_1dk_2\right),\end{aligned}$$

where $\delta(\chi)=1$ *if* $\chi|_{\mathcal{O}_F^\times}=1$ *and* $\delta(\chi)=0$ *if* $\chi|_{\mathcal{O}_F^\times}\neq 1$.

Proof. For $S\in\mathcal{B}_P(\chi)$, $\Psi_S\in\mathcal{A}_\chi(M,(\tau_\Gamma)_{P|P})$ and $c(1,\lambda)\Psi_S, c(w,\lambda)\Psi_S\in\mathcal{A}_\chi(M,(\tau_\Gamma)_{P|\overline{P}})$. Therefore $\Psi=\Psi_S$ can be written as a sum of matrix coefficients of χ. Thus

$$\begin{aligned}&(C^P E_P)(m,\Psi,\lambda)_{k_1,k_2}\\ =&(c(1,\lambda)\Psi)(m)_{k_1,k_2}e^{\lambda H_M(m)}+(c(w,\lambda)\Psi)(m)_{k_1,k_2}e^{-\lambda H_M(m)}\\ =&\chi(m)[(c(1,\lambda)\Psi)(1)_{k_1,k_2}e^{\lambda H_M(m)}+(c(w,\lambda)\Psi)(1)_{k_1,k_2}e^{-\lambda H_M(m)}]\end{aligned}$$

where $\chi(m)\in\mathbb{C}^\times$. Hence

$$\begin{aligned}&\int_{\widetilde{M_H}^+}D_{P_H}(m)\delta_P^{-1/2}(m)(c(1,\lambda)\Psi)(1)_{k_1,k_2}e^{(\lambda+\nu)(H_M(m))}\chi(m)(1-u(m,t))dm\\ &+\int_{\widetilde{M_H}^+}D_{P_H}(m)\delta_P^{-1/2}(m)(c(w,\lambda)\Psi)(1)_{k_1,k_2}e^{(-\lambda+\nu)(H_M(m))}\chi(m)(1-u(m,t))dm\\ =&(1+q^{-1})(c(1,\lambda)\Psi)(1)_{k_1,k_2}\int_{\widetilde{M_H}^+}e^{(\lambda+\nu)(H_M(m))}\chi(m)(1-u(m,t))dm\\ &+(1+q^{-1})(c(w,\lambda)\Psi)(1)_{k_1,k_2}\int_{\widetilde{M_H}^+}e^{(-\lambda+\nu)(H_M(m))}\chi(m)(1-u(m,t))dm\\ =&(1+q^{-1})\int_{\mathcal{O}_F^\times}\chi(\alpha)d^\times\alpha\sum_{n=t+1}^{\infty}[(c(1,\lambda)\Psi)(1)_{k_1,k_2}q^{2(\lambda+\nu)n}+(c(w,\lambda)\Psi)(1)_{k_1,k_2}q^{2(-\lambda+\nu)n}].\end{aligned}$$

Clearly $\int_{\mathcal{O}_F^\times}\chi(\alpha)d^\times\alpha=0$ unless $\chi|_{\mathcal{O}_F^\times}=1$. If $\chi|_{\mathcal{O}_F^\times}=1$, the previous line equals

$$\delta(\chi)(1+q^{-1})\left[\frac{q^{2(\lambda+\nu)(t+1)}}{1-q^{2(\lambda+\nu)}}c(1,\lambda)\Psi(1)_{k_1,k_2}+\frac{q^{2(-\lambda+\nu)(t+1)}}{1-q^{2(-\lambda+\nu)}}c(w,\lambda)\Psi(1)_{k_1,k_2}\right].$$

Hence

$$\int^{\sharp}_{\widetilde{M}_H^+(F)} D_{P_H}(m)\delta_P^{-1/2}(m)C^P E_P(m,\Psi_S,\lambda)_{k_1,k_2}(1-u(m,t))dm$$
$$=\delta(\chi)(1+q^{-1})\left(\frac{q^{2\lambda(t+1)}}{1-q^{2\lambda}}c(1,\lambda)\Psi_S(1)_{k_1,k_2}+\frac{q^{-2\lambda(t+1)}}{1-q^{-2\lambda}}c(w,\lambda)\Psi_S(1)_{k_1,k_2}\right)$$

and the proposition now follows. □

Lemma 4.8. *Let* $\chi=(\chi,\chi^{-1})$. *Then*

(1) *If* $\chi|_{F^\times}\neq 1$ *and* $\chi|_{E^1}\neq 1$, *then*

$$\int^*_{\widetilde{H}(F)}\operatorname{tr}(I_P(\chi_\lambda,h)S)dh=\int_{\widetilde{H}(F)}\operatorname{tr}(I_P(\chi_\lambda,h)u(h,t)dh=0.$$

(2) *If* $\chi|_{F^\times}\neq 1$ *and* $\chi|_{E^1}=1$, *then for* $t\gg 0$,

$$\int^*_{\widetilde{H}(F)}\operatorname{tr}(I_P(\chi_\lambda,h)S)dh=\int_{\widetilde{H}(F)}\operatorname{tr}(I_P(\chi_\lambda,h)S)u(h,t)dh.$$

(3) *If* $\chi|_{F^\times}=1$ *and* $\chi|_{E^1}\neq 1$, *then* $\int^*_{\widetilde{H}}\operatorname{tr}(I_P(\chi_\lambda,h)S)dh$ *is* 0 *whenever defined and*

$$\int_{\widetilde{K}_H\times\widetilde{K}_H}c(1,\lambda)\Psi(1)_{k_1,k_2}dk_1dk_2=\int_{\widetilde{K}_H\times\widetilde{K}_H}c(s,\lambda)\Psi(1)_{k_1,k_2}dk_1dk_2$$

at $\lambda=0$.

(4) *If* $\chi|_{F^\times}=1$ *and* $\chi|_{E^1}=1$, *then* $\chi^2=1$. *In this case* $c(1,\lambda)$ *and* $c(s,\lambda)$ *have a simple pole at* $\lambda=0$ *and so* $\mu(\chi_\lambda)$ *has a zero of order two at* $\lambda=0$ *and* $\mu(\chi_\lambda)c(1,\lambda)=\mu(\chi_\lambda)c(s,\lambda)=0$ *at* $\lambda=0$.

In all cases,

$$\mu(\chi_\lambda)\int^*_{\widetilde{H}(F)}\operatorname{tr}(I_P(\chi_\lambda,h)S)dh$$

is holomorphic for all $\lambda\in i\mathbb{R}$, $S\in\mathcal{B}_P(\chi)$.

Proof. Case 2 is obvious from the above work. Case 1 is obvious from the above work and the H invariance of $\int^*_{\widetilde{H}}\operatorname{tr}(I_P(\chi_\lambda,h)S)dh$.

The vanishing of the regularized period in case 3 also follows from H-invariance. Then by the previous proposition we know that for $\lambda\neq 0$,

$$\int_{\widetilde{H}(F)}\operatorname{tr}(I_P(\chi_\lambda,h)S)u(h,t)dh$$
$$=-(1+q^{-1})\left(\frac{q^{2\lambda(t+1)}}{1-q^{2\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H}c(1,\lambda)\Psi_S(1)_{k_1,k_2}dk_1dk_2+\frac{q^{-2\lambda(t+1)}}{1-q^{-2\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H}c(w,\lambda)\Psi_S(1)_{k_1,k_2}dk_1dk_2\right).$$

Both sides are holomorphic and the left hand side is also defined and holomorphic for $\lambda=0$. As

$$\operatorname{Res}_{\lambda=0}\frac{q^{2\lambda(t+1)}}{1-q^{2\lambda}}=\frac{-1}{2\log q}$$

and

$$\operatorname{Res}_{\lambda=0}\frac{q^{-2\lambda(t+1)}}{1-q^{-2\lambda}}=\frac{1}{2\log q}$$

we must have that

$$\int_{\widetilde{K}_H\times\widetilde{K}_H}c(1,0)\Psi(1)_{k_1,k_2}dk_1dk_2=\int_{\widetilde{K}_H\times\widetilde{K}_H}c(w,0)\Psi(1)_{k_1,k_2}dk_1dk_2.$$

In case 4 the poles and zeros are well known and can also be seen by explicit computations of the intertwining operators. We have that

$$\begin{aligned}
&\mu(\chi_\lambda)\int_{\widetilde{H}(F)}\mathrm{tr}(I_P(\chi_\lambda,h)S)u(h,t)dh\\
=&\mu(\chi_\lambda)\int^*_{\widetilde{H}(F)}\mathrm{tr}(I_P(\chi_\lambda,h)S)dh\\
&-(1+q^{-1})\mu(\chi_\lambda)\left(\frac{q^{2\lambda(t+1)}}{1-q^{2\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H}c(1,\lambda)\Psi_S(1)_{k_1,k_2}dk_1dk_2+\frac{q^{-2\lambda(t+1)}}{1-q^{-2\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H}c(w,\lambda)\Psi_S(1)_{k_1,k_2}dk_1dk_2\right).
\end{aligned}$$

The left hand side is 0 at $\lambda=0$ and the last two terms are holomorphic at $\lambda=0$ so the first term must be holomorphic at $\lambda=0$.

□

Let

$$D_{\chi_\lambda}(f)=\sum_{S'\in\mathcal{B}_P(\chi)}P_{\chi_\lambda}(S_{\chi_\lambda}[f])\overline{W_{\chi_\lambda}(S)},$$

$$P_{\chi_\lambda}(S)=\int^*_{\widetilde{H}(F)}\mathrm{tr}(I_P(\chi_\lambda,h)S)u(h,t)dh$$

and

$$\widetilde{D}_\chi(f)=(1+q^{-1})\mu(\chi_0)\sum_{S\in\mathcal{B}_P(\chi)}\left[\iint_{\widetilde{K}_H\times\widetilde{K}_H}c(1,0)\psi_{S_0[f]}(1)_{k_1,k_2}dk_1dk_2\right]W_{\chi_0}(S).$$

We now relate the distributions above to the truncated distributions from (4.2).

Lemma 4.9. *Let $\chi=(\chi,\chi^{-1})$.*

(1) *If $\chi|_{F^\times}\neq 1$ and $\chi|_{E^1}\neq 1$, then*

$$\lim_{t\to\infty}\int_0^{\frac{\pi i}{\log q}}\mu(\chi_\lambda)D^t_{\chi_\lambda}(f)d\lambda=0.$$

(2) *If $\chi|_{F^\times}\neq 1$ and $\chi|_{E^1}=1$, then*

$$\lim_{t\to\infty}\int_0^{\frac{\pi i}{\log q}}\mu(\chi_\lambda)D^t_{\chi_\lambda}(f)d\lambda=\int_0^{\frac{\pi i}{\log q}}\mu(\chi_\lambda)D_{\chi_\lambda}(f)d\lambda.$$

(3) *If $\chi|_{F^\times}=1$ and $\chi|_{E^1}\neq 1$, then*

$$\lim_{t\to\infty}\int_0^{\frac{\pi i}{\log q}}\mu(\chi_\lambda)D^t_{\chi_\lambda}(f)d\lambda=\widetilde{D}_\chi(f).$$

(4) *If $\chi|_{F^\times}=1$ and $\chi|_{E^1}=1$, then*

$$\lim_{t\to\infty}\int_0^{\frac{\pi i}{\log q}}\mu(\chi_\lambda)D^t_{\chi_\lambda}(f)d\lambda=\int_0^{\frac{\pi i}{\log q}}\mu(\chi_\lambda)D_{\chi_\lambda}(f)d\lambda.$$

Proof. First we note that

$$\int_0^{\frac{\pi i}{\log q}}\mu(\chi_\lambda)D^t_{\chi_\lambda}(f)d\lambda=\int_0^{\frac{\pi i}{\log q}}\mu(\chi_\lambda)\sum_{S\in\mathcal{B}_P(\chi)}P^t_{\chi_\lambda}(S[f])\overline{W_{\chi_\lambda}(S)}d\lambda$$

$$=\sum_{S\in\mathcal{B}_P(\chi)}\int_0^{\frac{\pi i}{\log q}}\int_{N(F)}\mathrm{tr}(I_P(\chi_\lambda,n)S)\psi(n)u(n,t)dn\mu(\chi_\lambda)\int_{\widetilde{H}(F)}\mathrm{tr}(I_P(\chi,h)S)u(h,t)dh.$$

By Proposition 4.7 for $t \gg 0$,

(4.5)

$$\int_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda, h)S)u(h,t)dh$$
$$= \int^*_{\widetilde{H}(F)} \operatorname{tr}(I_P(\chi_\lambda, h)S)dh +$$
$$\delta(\chi)(1+q^{-1})\left(\frac{q^{2\lambda(t+1/2)}}{q^\lambda - q^{-\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H} c(1,\lambda)\Psi(1)_{k_1,k_2}dk_1dk_2 + \frac{-q^{-2\lambda(t+1/2)}}{q^\lambda - q^{-\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H} c(w,\lambda)\Psi(1)_{k_1,k_2}dk_1dk_2\right).$$

Cases 1 and 2 now follow directly from Lemma 4.8. In case 3, by Lemma 4.8 the regularized period vanishes and we are left computing

$$(1+q^{-1})\lim_{t\to\infty}\int_0^{\frac{\pi i}{\log q}} \mu(\chi_\lambda)\overline{W_{\chi_\lambda}(S)}\Big(\frac{q^{2\lambda(t+1)}}{1-q^{2\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H} c(1,\lambda)\Psi(1)_{k_1,k_2}dk_1dk_2$$
$$+\frac{q^{-2\lambda(t+1)}}{1-q^{-2\lambda}}\int_{\widetilde{K}_H\times\widetilde{K}_H} c(w,\lambda)\Psi(1)_{k_1,k_2}dk_1dk_2\Big)d\lambda. \tag{4.6}$$

Let

$$f_1(\lambda) = \frac{1+q^{-1}}{2}\mu(\chi_\lambda)\overline{W_{\chi_\lambda}(S)}\left(\int_{\widetilde{K}_H\times\widetilde{K}_H} c(1,\lambda)\Psi(1)_{k_1,k_2}dk_1dk_2 - \int_{\widetilde{K}_H\times\widetilde{K}_H} c(w,\lambda)\Psi(1)_{k_1,k_2}dk_1dk_2\right)$$

and

$$f_2(\lambda) = \frac{1+q^{-1}}{2}\mu(\chi_\lambda)\overline{W_{\chi_\lambda}(S)}\left(\int_{\widetilde{K}_H(F)\times\widetilde{K}_H(F)} c(1,\lambda)\Psi(1)_{k_1,k_2}dk_1dk_2 + \int_{\widetilde{K}_H(F)\times\widetilde{K}_H(F)} c(w,\lambda)\Psi(1)_{k_1,k_2}dk_1dk_2\right).$$

Then (4.6) equals

$$\lim_{t\to\infty}\int_0^{\frac{\pi i}{\log q}} \frac{f_1(\lambda)(q^{2\lambda(t+\frac12)}+q^{-2\lambda(t+\frac12)})}{q^\lambda - q^{-\lambda}} + \lim_{t\to\infty}\int_0^{\frac{\pi i}{\log q}} \frac{f_2(\lambda)(q^{2\lambda(t+\frac12)}-q^{-2\lambda(t+\frac12)})}{q^\lambda - q^{-\lambda}}d\lambda.$$

By Lemma 4.8, $f_1(0) = 0$. Hence by Fourier analysis the first integral will vanish. The limit of the second integral will be $f_2(0)$, which, by the identity in case 3 of Lemma 4.8 equals

$$(1+q^{-1})\mu(\chi_0)\overline{W_{\chi_0}(S)}\int_{\widetilde{K}_H\times\widetilde{K}_H} c(1,0)\Psi(1)_{k_1,k_2}dk_1dk_2.$$

For case 4 by Lemma 4.8 when multiplied by $\mu(\chi_\lambda)\overline{W_{\chi_\lambda}(S)}$, $f_1(\lambda)$ and $f_2(\lambda)$ are holomorphic functions of λ and vanish at $\lambda = 0$, thus by similar analysis as above the last two terms vanish in the limit and we are left with the statement of the Lemma. □

4.2.1. *Discrete series representations.* Because the matrix coefficient of a supercuspidal representation σ has compact support, it is obvious by the definition of the regularized integral that

$$\lim_{t\to\infty}\int_{\widetilde{H}(F)} \operatorname{tr}(\sigma(h)S)u(h,t)dh = \int_{\widetilde{H}(F)} \operatorname{tr}(\sigma(h)S)dh.$$

Now we will prove that this is also true for Steinberg representations.

Lemma 4.10. *For $\sigma = St(\chi), \chi^2 = 1$, the matrix coefficients are absolutely convergent over $\widetilde{H}$. Thus the limit*

$$\lim_{t\to\infty}\int_{\widetilde{H}(F)} \operatorname{tr}(\sigma(h)S)u(h,t)dh$$

exists and equals

$$\int_{\widetilde{H}(F)} \operatorname{tr}(\sigma(h)S)dh.$$

Proof. By [BW80, XI.4.3] and [Cas, 4.2.3], a matrix coefficient for σ evaluated at $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is equal to a matrix coefficient for the Jacquet functor σ_N, evaluated at the same value, for $\left|\frac{a}{b}\right|_E$ sufficiently small. The Jacquet functor of σ is δ_P. Thus outside some ball of compact support, our original matrix coefficient will behave like δ_P on M_H^-. When we integrate over $\widetilde{H}(F)$, using the $K_H M_H^- K_H$ decomposition, we get a measure factor of $\delta_P^{-1/2}$. Thus outside a ball of compact support our integral will look like

$$\int_{|a|<c} |a|_F d^\times a$$

for some $c > 0$. □

Putting everything together we have proved the following.

Proposition 4.11. *For any* $f \in C_c^\infty(\widetilde{G}(F) \times \widetilde{G}(F))$,

$$\lim_{t\to\infty} \int_{\widetilde{H}(F)} \int_{N(F)} K_f(h,n)\psi(n)u(h,t)u(n,t)dndh$$

$$= \sum_{\sigma\in\Pi_2(\widetilde{G}(F))} d(\sigma)D_\sigma(f) + \frac{1}{2} \sum_{\substack{\chi\in\{\Pi_2(\widetilde{M}(F))\} \\ \chi^2\neq 1, \chi|_{F^\times}=1}} \widetilde{D}_\chi(f) + \frac{1}{2} \sum_{\substack{\chi\in\{\Pi_2(\widetilde{M}(F))\} \\ \chi|_{E^1}=1}} \int_0^{\frac{\pi i}{\log q}} \mu(\chi_\lambda) D_{\chi_\lambda}(f) d\lambda,$$

where

$$D_{\chi_\lambda}(f) = \sum_{S\in\mathcal{B}_P(\chi)} P_{\chi_\lambda}(S_\lambda[f])\overline{W_{\chi_\lambda}(S)},$$

$$P_{\chi_\lambda}(S) = \int_{\widetilde{H}(F)}^* \mathrm{tr}(I_P(\chi_\lambda, h)S)dh,$$

$$\widetilde{D}_\chi(f) = (1+q^{-1})\mu(\chi_0) \sum_{S\in\mathcal{B}_P(\chi)} \left[\int_{\widetilde{K}_H(F)\times\widetilde{K}_H(F)} c(1,0)\psi_{S_0[f]}(1)_{k_1,k_2} dk_1 dk_2\right] W_{\chi_0}(S),$$

$$D_\sigma(f) = \sum_{S\in\mathcal{B}(\sigma)} P_\sigma(\sigma(f_2)S\sigma(f_1^\vee))\overline{W_\sigma(S)},$$

and

$$P_\sigma(S) = \int_{\widetilde{H}(F)} \mathrm{tr}(\sigma(h)S)dh.$$

The result of this proposition, combined with Proposition 4.3, proves Theorem 1.4.

5. Comparison of local trace formulas and applications

We now put together the results of the last two sections to compare the two trace formulas.

Definition 5.1. *We say that* $f' \in C_c^\infty(\widetilde{G}'(F))$ *and* $f \in C_c^\infty(\widetilde{G}(F))$ *are matching functions if*

$$O'(f', \psi', a) = O(f, \psi, a)$$

for all $a \in E^\times$.

By work of Flicker [Fli91] and Ye [Ye89], we know that for any $f' \in C_c^\infty(\widetilde{G}'(F))$ there exists a matching $f \in C_c^\infty(\widetilde{G}(F))$ and vice versa. In fact, by the Fundamental Lemma, for f' spherical, we know that f is the corresponding function from the base change map between their Hecke algebras. Thus by the geometric expansion of the trace formulas in Propositions 3.4 and 4.3 we have the following.

Proposition 5.2. *For* $f_i' \in C_c^\infty(\widetilde{G}'(F))$ *and* $f_i \in C_c^\infty(\widetilde{G}(F))$ *matching functions for* $i = 1, 2$,

$$\lim_{t\to\infty} \int_{N'(F)} \int_{N'(F)} K_{f_1'\otimes f_2'}(n_1, n_2)\psi'(n_1^{-1}n_2)u(n_1,t)u(n_2,t)dn_1dn_2$$

$$= \lim_{t\to\infty} \int_{\widetilde{H}(F)} \int_{N(F)} K_{f_1\otimes f_2}(h,n)\psi(n)u(h,t)u(n,t)dndh.$$

Now we use the equality of the trace formulas to compare the spectral expansions. By Propositions 3.6, 4.11 and 5.2,

Theorem 5.3. *For f_i and f_i' matching functions for $i = 1, 2$,*

$$\sum_{\sigma' \in \Pi_2(\widetilde{G}'(F))} d(\sigma') D'_{\sigma'}(f_1' \otimes f_2') + \frac{1}{2} \sum_{\chi' \in \{\Pi_2(\widetilde{M}'(F))\}} d(\chi') \int_0^{\frac{\pi i}{\log q}} \mu(\chi'_\lambda) D'_{\chi'_\lambda}(f_1' \otimes f_2') d\lambda$$

$$= \sum_{\sigma \in \Pi_2(\widetilde{G}(F))} d(\sigma) D_\sigma(f_1 \otimes f_2) + \frac{1}{2} \sum_{\substack{\chi \in \{\Pi_2(\widetilde{M}(F))\} \\ \chi^2 \neq 1, \chi|_{F^\times} = 1}} \widetilde{D}_\chi(f_1 \otimes f_2)$$

$$+ \frac{1}{2} \sum_{\substack{\chi \in \{\Pi_2(\widetilde{M}(F))\} \\ \chi|_{E^1} = 1}} d(\chi) \int_0^{\frac{\pi i}{\log q}} \mu(\chi_\lambda) D_{\chi_\lambda}(f_1 \otimes f_2) d\lambda.$$

The nonstandard base change map lifts principal series representations of $\widetilde{G}'$ to principal series representations $I_P(\chi)$ of $\widetilde{G}$ such that $\chi|_{E^1} = 1$. It also lifts certain square integrable representations of $\widetilde{G}'$ to the principal series representations of $\widetilde{G}$ defined by $I_P(\chi)$ such that $\chi^2 \neq 1, \chi|_{F^\times} = 1$. It lifts the remaining square integrable representations of $\widetilde{G}'$ to square integral representations of $\widetilde{G}$ [Rog90]. Thus we could rephrase the right hand side of Theorem 5.3 in terms of summing over the representations of $\widetilde{G}$ that are the nonstandard base change lifts of representations of $\widetilde{G}'$. The extra discrete term $\widetilde{W}_\chi(f)$ corresponds exactly to the representations that lift from the discrete series of $\widetilde{G}'$ to the principal series of $\widetilde{G}$.

We also note that the only representations that appear on the right hand side of Theorem 5.3 are those σ or $I_P(\chi_\lambda)$ for which there is a matrix coefficient such that the regularized integral over H is non-zero. This gives us a more explicit description of the nonvanishing H invariant linear functional that characterizes the image of the nonstandard base change map.

We would like to relate our distributions to the local factors in the Bessel and relative Bessel distributions. Recall from the introduction that Jacquet's global relative trace formula tells us that for $f' \in C_c^\infty(U(2, \mathbb{A}_F))$ and $f \in C_c^\infty(\mathrm{GL}(2, \mathbb{A}_E))$ matching functions, if a cuspidal representation π' of $U(2, \mathbb{A}_F)$ maps to π of $\mathrm{GL}(2, \mathbb{A}_E)$ under nonstandard base change, then

$$B'_{\pi'}(f') = B_\pi(f)$$

where

$$B'_{\pi'}(f') = \sum_{\phi' \in \text{o.n.b.}(V_{\pi'})} W'(\pi'(f')\phi') \overline{W'(\phi')},$$

$$B_\pi(f) = \sum_{\phi \in \text{o.n.b.}(V_\pi)} P(\pi(f)\phi) \overline{W(\phi)},$$

$$W'(\phi') = \int_{N'(F) \backslash N'(\mathbb{A}_F)} \phi'(n) \overline{\psi'(n)} dn,$$

$$W(\phi) = \int_{N(E) \backslash N(\mathbb{A}_E)} \phi(n) \overline{\psi(n)} dn,$$

and

$$P(\phi) = \int_{\mathrm{GL}(2,F) \backslash \mathrm{GL}(2, \mathbb{A}_F)} \phi(h) dh \neq 0.$$

While $B'_{\pi'}(f')$ and $B_\pi(f)$ factor into local Bessel distributions $B'_{\pi'_v}(f'_v)$ and $B_{\pi_v}(f_v)$, it is not clear how to normalize the local Bessel distributions. We can rewrite our local distributions as a product of two local Bessel (or local relative Bessel) distributions:

Lemma 5.4. (1) *For σ' an irreducible supercuspidal representation of $\widetilde{G}'$, there exists a local Bessel distribution $B'_{\sigma'}$, that is unique up to a constant of absolute value 1, such that*

$$D'_{\sigma'}(f_1' \otimes f_2') = B'_{\sigma'}(f_2') B'_{\sigma'^*}(f_1').$$

(2) *For σ an irreducible supercuspidal representation of $\widetilde{G}$, there exists a local relative Bessel distribution B_σ, that is unique up to a constant of absolute value 1, such that*

$$D_\sigma(f_1 \otimes f_2) = B_\sigma(f_2) B_{\sigma^*}(f_1).$$

Proof. We recall that

$$D'_{\sigma'}(f') = \sum_{S' \in \mathcal{B}(\sigma')} \int_{N'(F)} \mathrm{tr}(\sigma'(n_1)\sigma'(f'_2)S'\sigma'^*(f'_1))\psi'(n_1)^{-1}dn_1 \overline{\int_{N'(F)} \mathrm{tr}(\sigma'(n_2)S')\psi'(n_2)^{-1}dn_2}.$$

Let $V = V_{\sigma'}$. As S' in an endomorphism on V there exist $v \in V, v^* \in V^*$ such that $S' = v \otimes v^*$. Then the linear functional on $V \otimes V^*$ that acts by

$$v \otimes v^* \mapsto \int_{N'(F)} \mathrm{tr}(\sigma'(n)v \otimes v^*)\psi'(n)^{-1}dn$$

transforms under n on v and v^* by ψ'. Thus it is a Whittaker functional on $V \otimes V^*$. By the uniqueness of Whittaker models,

$$\int_{N'(F)} \mathrm{tr}(\sigma'(n)S')\psi'(n)^{-1}dn = W'(v)W'(v^*).$$

Thus

$$\begin{aligned} D'_{\sigma'}(f') = &\sum_{v \otimes v^*} W'(\sigma'(f'_2)v)\overline{W'(v)}W'(\sigma'^*(f'_1)v^*)\overline{W'(v^*)} \\ =&B'_{\sigma'}(f'_2)B'_{\sigma'^*}(f'_1). \end{aligned}$$

We note that if we change $B'_{\sigma'}$ by a constant c, then $B'_{\sigma'^*}$ will change by $\overline{c}$.

The proof for the local relative Bessel distributions is similar, using the uniqueness of the H-invariant linear functional. ☐

These results allow us to describe matching functions by an equality of all the Bessel distributions.

Lemma 5.5 (Density)**.** (1) *If $f'_1 \in C_c^\infty(\widetilde{G}'(F))$ is such that $B'_{\sigma'}(f'_1) = 0$ for all irreducible tempered representations σ' of $\widetilde{G}'$, then $O'(f'_1, \psi'^{-1}, a) = 0$ for all $a \in E^\times$.*

(2) *If $f_1 \in C_c^\infty(\widetilde{G}(F))$ is such that $B_\sigma(f_1) = 0$ for all irreducible tempered representations σ of $\widetilde{G}$, then $O(f_1, \psi^{-1}, a) = 0$ for all $a \in E^\times$.*

Proof. If $B'_{\sigma'}(f'_1) = 0$ for all σ', then by Theorem 1.3 and Lemma 5.4,

$$\int_{a \in E^\times/E^1} |a| O'(f'_1, \psi'^{-1}, a) O'(f'_2, \psi', a) d^\times a = 0$$

for all $f'_2 \in C_c^\infty(\widetilde{G}')$. As $O'(f'_1, \psi^{-1}, a)$ is a locally constant function of a there exists some open compact U such that $O'(f'_1, \psi'^{-1}, a)$ is bi-invariant under it. Then by choosing f'_2 such that $O'(f'_2, \psi'^{-1}, a)$ has support contained in U we see that $O'(f'_1, \psi'^{-1}, a) = 0$. The second case follows from the first one. ☐

Combining Theorem 5.3 with Lemma 5.4 and the global relative trace formula, we have the following result:

Corollary 5.6. *If σ is the nonstandard base change lift of the supercuspidal representation σ', and f'_i and f_i are matching functions for $i = 1, 2$, then*

$$d(\sigma')D_{\sigma'}(f'_1 \otimes f'_2) = d(\sigma)D_\sigma(f_1 \otimes f_2).$$

Proof. From the global comparison of relative trace formulas [Fli91, Lap06, Ye89], a standard globalization argument and Lemma 5.4, we know there exists a constant c_σ such that $D'_{\sigma'}(f') = c_\sigma D_\sigma(f)$ for all matching f, f'. Take f'_1 and f'_2 to be matrix coefficients of σ' such that $B'_{\sigma'}(f'_i) \neq 0$. Take f_1 to be a matching function of f'_1. Then $B_\pi(f_1) = cB'_{\pi'}(f'_1) = 0$ unless $\pi' \sim \sigma'$. Let $Pr_\sigma(f_1)$ denote the projection of f_1 to σ. Then $B_\pi(Pr_\sigma(f_1)) = 0$ if $\pi \not\sim \sigma$ and $B_\sigma(Pr_\sigma(f_1)) = B_\sigma(f_1)$. Thus by the density lemma above, f_1 and $Pr_\sigma(f_1)$ have identical orbital integrals. As f_1 is a matching function for f'_1 then $Pr_\sigma(f_1)$ must be a matching function for f'_1 as well. Take f_2 a matching function to f'_2. Then by Theorem 5.3,

$$d(\sigma')D'_{\sigma'}(f') = d(\sigma)D_\sigma(f)$$

so $c = \frac{d(\sigma)}{d(\sigma')}$. □

In addition to the spectral comparison, these local trace formulas also have applications on the geometric side. If we define the inner product of two functions g_1, g_2 on $E^\times/E^1$ by

$$\langle g_1, g_2 \rangle = \int_{a \in E^\times/E^1} |a|_E g_1(a) g_2(a) d^\times a,$$

then

Corollary 5.7 (Orthogonality Relations)**.** *For f_1 (resp. f_1') and f_2 (resp. f_2') matrix coefficients of the supercuspidal representations σ_1 (resp. σ_1') and σ_2 (resp. σ_2'),*

$$\langle O'\left(f_1', \psi', \cdot\right), O'\left(f_2', \psi'^{-1}, \cdot\right)\rangle \neq 0 \textit{ iff } \sigma_1' \sim \sigma_2'$$

and

$$\langle O\left(f_1, \psi, \cdot\right), O\left(f_2, \psi^{-1}, \cdot\right)\rangle \neq 0 \textit{ iff } \sigma_1 \sim \sigma_2.$$

Proof. This follows directly from the local Kuznetsov and local relative trace formulas. □

6. Acknowledgments

The author thanks Erez Lapid for his helpful suggestions in an early stage of this project and Omer Offen for his useful comments. Finally, the author thanks the referee for helpful comments.

References

[Art89] James Arthur, *Towards a local trace formula*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 1–23. MR MR1463695 (98h:22020)

[Art91] ———, *A local trace formula*, Inst. Hautes Études Sci. Publ. Math. (1991), no. 73, 5–96. MR MR1114210 (92f:22029)

[BW80] Armand Borel and Nolan R. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Annals of Mathematics Studies, vol. 94, Princeton University Press, Princeton, N.J., 1980. MR MR554917 (83c:22018)

[Cas] Bill Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*, unpublished notes.

[Fli91] Yuval Z. Flicker, *On distinguished representations*, J. Reine Angew. Math. **418** (1991), 139–172. MR MR1111204 (92i:22019)

[HC76] Harish-Chandra, *Harmonic analysis on real reductive groups. III. The Maass-Selberg relations and the Plancherel formula*, Ann. of Math. (2) **104** (1976), no. 1, 117–201. MR MR0439994 (55 #12875)

[HC84] ———, *The plancherel formula for reductive p-adic groups*, Collected papers. Vol. IV, Springer-Verlag, New York, 1984, pp. 353–367. MR MR726026 (85e:01061d)

[Jac05] Hervé Jacquet, *A guide to the relative trace formula*, Automorphic representations, L-functions and applications: progress and prospects, Ohio State Univ. Math. Res. Inst. Publ., vol. 11, de Gruyter, Berlin, 2005, pp. 257–272. MR MR2192826

[JL85] H. Jacquet and K. F. Lai, *A relative trace formula*, Compositio Math. **54** (1985), no. 2, 243–310. MR MR783512 (86j:11059)

[JLR99] Hervé Jacquet, Erez Lapid, and Jonathan Rogawski, *Periods of automorphic forms*, J. Amer. Math. Soc. **12** (1999), no. 1, 173–240. MR MR1625060 (99c:11056)

[Lap06] Erez Lapid, *On the fine spectral expansion of jacquet's relative trace formula*, J. Inst. Math. Jussieu **5** (2006), no. 2, 263–308.

[Rog90] Jonathan D. Rogawski, *Automorphic representations of unitary groups in three variables*, Annals of Mathematics Studies, vol. 123, Princeton University Press, Princeton, NJ, 1990. MR MR1081540 (91k:22037)

[Sil96] Allan J. Silberger, *Harish-Chandra's Plancherel theorem for p-adic groups*, Trans. Amer. Math. Soc. **348** (1996), no. 11, 4673–4686. MR MR1370652 (99c:22026)

[Spr85] T. A. Springer, *Some results on algebraic groups with involutions*, Algebraic groups and related topics (Kyoto/Nagoya, 1983), Adv. Stud. Pure Math., vol. 6, North-Holland, Amsterdam, 1985, pp. 525–543. MR MR803346 (86m:20050)

[Wal03] J.-L. Waldspurger, *La formule de Plancherel pour les groupes p-adiques (d'après Harish-Chandra)*, J. Inst. Math. Jussieu **2** (2003), no. 2, 235–333. MR MR1989693 (2004d:22009)

[Ye89] Yangbo Ye, *Kloosterman integrals and base change for GL(2)*, J. Reine Angew. Math. **400** (1989), 57–121. MR MR1013725 (90i:11134)

Department of Mathematics, The City College of New York, CUNY, NAC 8/133, New York, NY 10031, USA
E-mail address: bfeigon@ccny.cuny.edu